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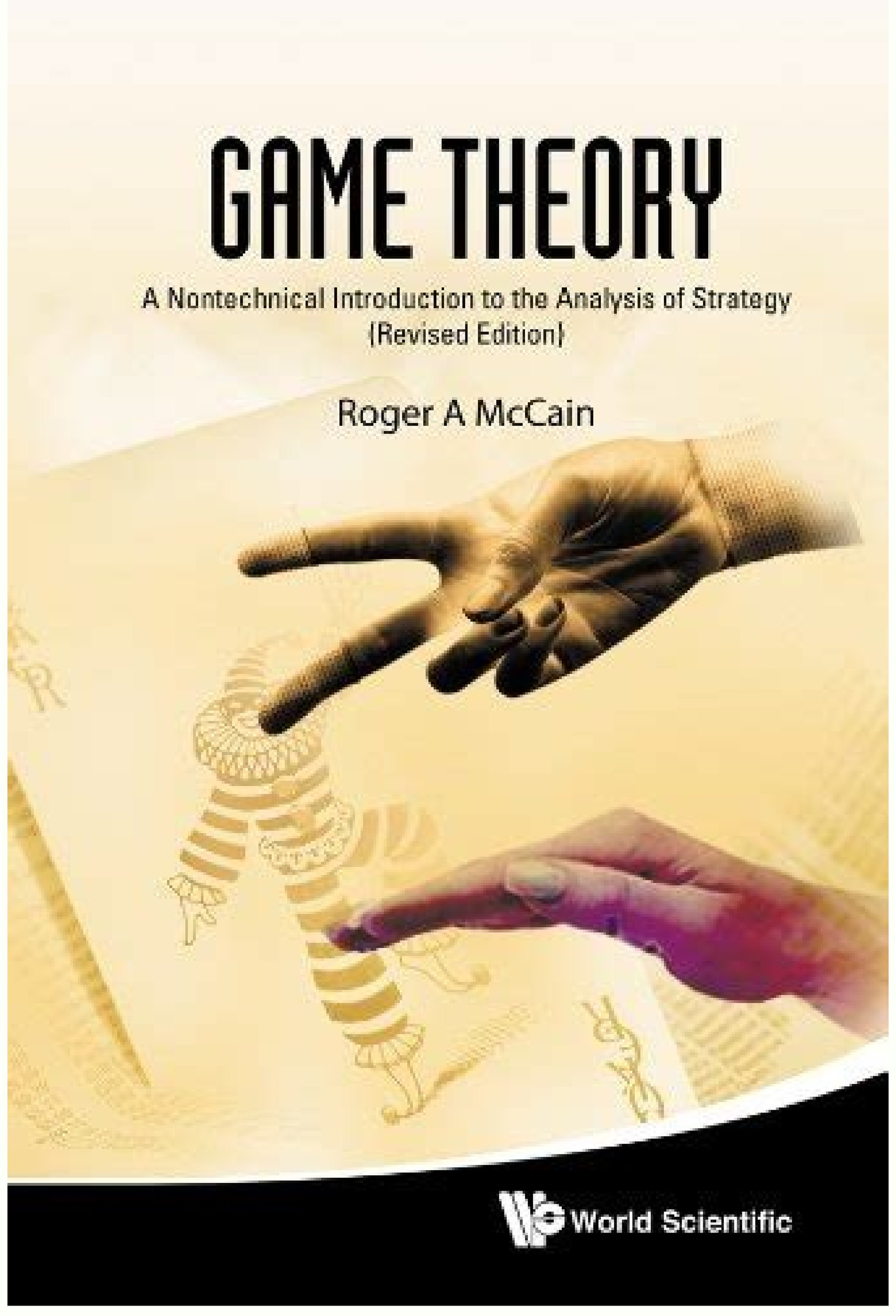
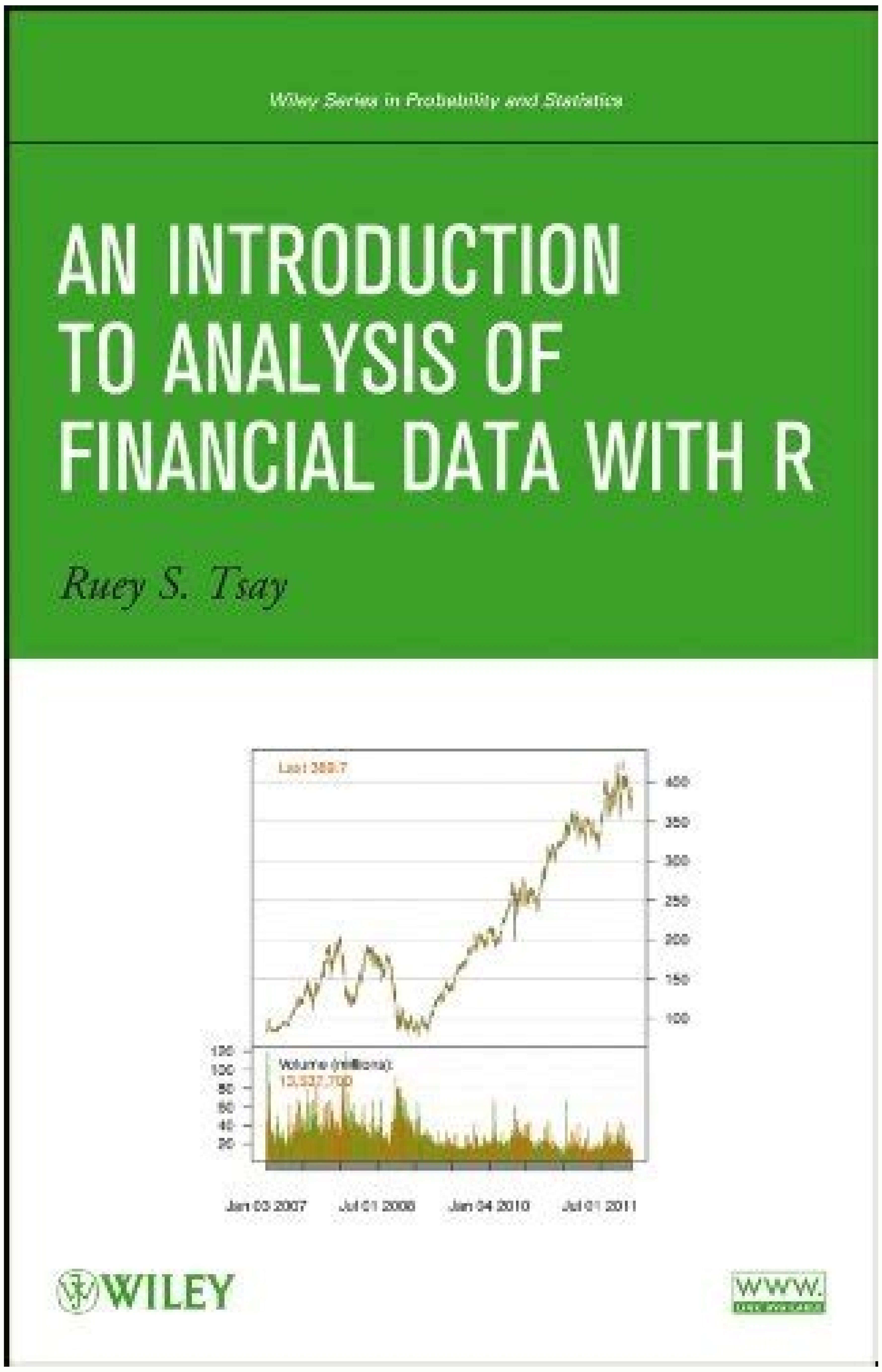




MGT 3510 COURSE OUTLINE AND ASSIGNMENT SCHEDULE SPRING 2015

Assignments listed below are firm only for *one week in advance*. The complete list of *tentative* assignments is given to help you plan ahead.

I. POSITIONING STRATEGIES IN MANUFACTURING AND SERVICES

Competitive Priorities *(Marketplace Drivers)*
Types of Technologies and Workforce Skills *(Resource Capabilities)*
Positioning Strategies *(Integration of Above)*

Assignment due Tues. Jan 13

1.0 Cheryl Gaimon, Karen Napoleon, "Information Technology: Worker Systems in Structured and Unstructured Environments," New Service Development: Creating Memorable Experiences, James A. Fitzsimmons and Mona Fitzsimmons, Eds., Sage Publications, Thousand Oaks CA, 2000, pp. 183-192, 212-214.

Reading due Thurs. Jan 15

Excerpts from Operations and Supply Chain Management, Jacobs and Chase, 14th Edition, 2014.

II. IMPROVING FIRM PERFORMANCE

A. New Product Development: Management and Engineering Interface

Assignment due Tues. Jan 20

Case 1: Campbell Soup *(Candidate for group write-up. See questions on syllabus)*
HBS Case 9-690-051, Aug. 1990.

Assignment due Thurs. Jan 22

1.25 Teradyne Corp.: The Jaguar Project (not a group case) HBS Case 9-606-042, May 2006.

Assignment due Tues. Jan 27

1.0 Behnam Tabrizi, Rick Walleigh, "Defining Next-Generation Products: An Inside Look," Harvard Business Review, Nov.-Dec. 1997.

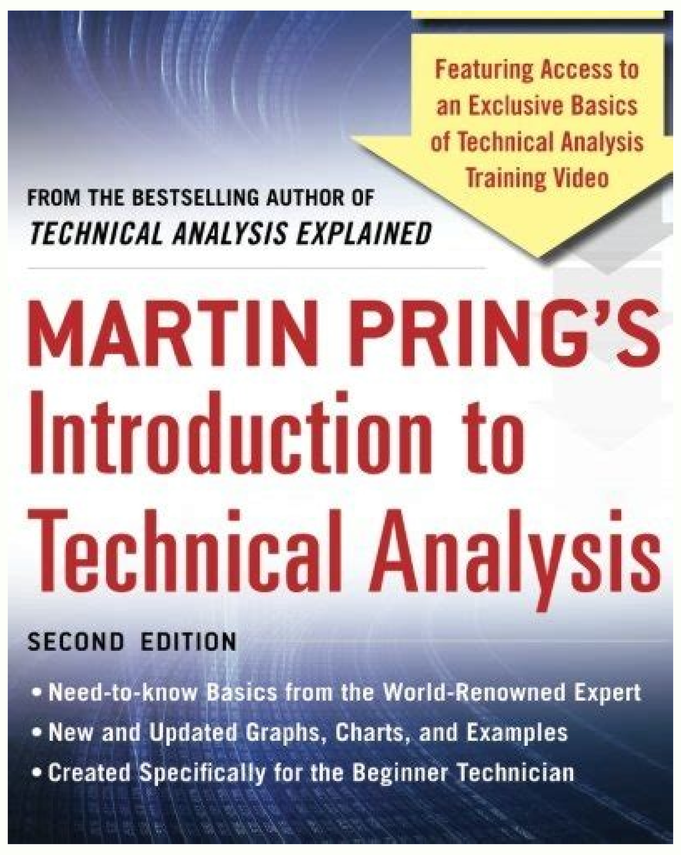
B. Manufacturing Flexibility

Assignment due Thurs. Jan 29

Case 2: Eli Lilly Company: The Flexible Facility Decision *(All groups must do this case and/or Universal Luxury)* ***HBS Case 9-694-074, April 1994.***

Assignment due Tues. Feb 3

1.25 Gaimon, Cheryl, Alysse Morton, "Investment in Changeover Flexibility for Early Entry in High Tech Markets," Production and Operations Management, Vol. 14, No. 2, 2005, 159-



Sammy Sample
9-17-10
Hour 1
Book Report
First Quarter

Title: The Adventures of Tom Sawyer
Author: Mark Twain
Genre: Realistic Fiction

Protagonist:
Tom Sawyer is the protagonist of the novel. He is about 12 years old and lives in the small town of St. Petersburg, Missouri. He wants to be an outlaw and likes to play hooky from school and church. He drives his Aunt Polly crazy and hates his half-brother Sid.

Antagonist:
Injun Joe is the antagonist of this story. Joe is a half breed Indian and a robber and murderer. He kills Dr. Robinson and in addition plans a brutal attack on a wealthy widow. He is greedy and bitter because of his treatment by the townspeople.

Conflict:
This is a person vs. person conflict. Tom wants to get the $12,000 that Joe has stolen and Joe wants to kill the Widow Douglas and then get away. Both characters cannot get what they want, so there is a conflict.

Setting:
The story takes place in a small town in Missouri during the 1830's. The setting is very important to the story because the town represents the place where Mark Twain grew up. Many of the characters are based on people he actually knew. The language, superstitions, and culture of the village are all important in this story.

Plot:
Tom Sawyer is a mischievous and hilarious boy growing up in the 1930's by the Mississippi River. He is constantly in trouble with Aunt Polly, who loves him and suffers trying to raise him. Tom is a very clever boy. He tricks his friends into whitewashing the fence, and wins a bible in church by cheating his friends out of their prize tickets.
Tom is not a good student and often plays hooky. He likes girls, and leaves Amy Lawrence for Becky Thatcher. His best friends are Joe Harper and Huck Finn. Huck is the son of the town drunk and disliked by all the adults. The kids respect him because he is the only free boy in town.

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If ϕ is the angle between (x, y) and B(x, y) then by (2) and what we just proved, cos ϕ = (x, y) · B(x, y) (x2 + y 2) cos θ = = cos θ. 4 b) E = {(x, y) : 0 ≤ y ≤ 1, y ≤ x ≤ 1} = {(x, y) : 0 ≤ x ≤ 1, 0 ≤ y ≤ x}, hence by Fubini's Theorem, Z 1 Z 0 Z 1 1 sin(x2) dx dy = y Z 0 x Z sin(x2) dy dx = 0 1 x sin(x2) dx = 0 1 − cos(1) . 10.5.4. a) If E is connected in R then E is an interval, hence E o is either empty or an interval, hence connected by definition or Theorem 10.56. Notice that x1 = 1 = y1 . Fix j ≥ n. By the Heine-Borel Theorem, xk is bounded, hence (by Bolzano–Weierstrass) has a convergent subsequence. Thus the trace looks like a sine wave traced on the parabolic cylinder y = x2 . [0, π]∪[2π, 3π]∪. c) limx→0 log(x/ sin x)/x2 = limx→0 (sin x − x cos x)/(2x2 sin x) = limx→0 x sin x/(2x2 cos x + 4x sin x) = limx→0 sin x/(2x cos x + 4 sin x) = limx→0 cos x/(6 cos x − 2x sin x) = 1/6. , QN such that ÃN !0 [∂E ∪ E0 ⊂ U := Qk k=1 PN and k=1 |Qk | < ^{2}/(4C), where |f (x)| ≤ C for x ∈ E. 2.2.6. By the Density of Rationals, there is an rn between x + 1/n and x for each n ∈ N. But by the Fundamental Theorem of Calculus and the fact that f (a) = 0, we have Z x f 0 (t) dt = f (x) − f (a) = f (x) a as required. Therefore, f is bounded on I ⊆ [a, b]. Pn Pn 6.4.8. By a sum angle formula and telescoping, we see that 2 sin(x/2) k=1 cos(kx) = k=1 (sin((k − 1/2)x) − sin((k + 1/2)x)) = sin(x/2) − sin((n + 1/2)x). It follows that ∂(y, z) ∂(z, x) + (yu ut + yv vt) ∂(u, v) ∂(u, v) ∂(x, y) + (zu ut + zv vt) ∂(u, v) ⌈ ⌉⌈ ⌉ xu yu zu xv yv zv = ut det ⌊ xu yu zu ⌋ + vt det ⌊ xu yu zu ⌋ = 0. b) By hypothesis, given ε > 0 there is an N ∈ N such that n ≥ N implies xn > 1/2 and |xn − 1| < ε/4. Hence by the Mean Value Theorem, |f (x + h) − f (x − h)| ≤ |f (x + h) − f (x)| + |f (x) − f (x − h)| = |h| |f 0 (c1)| + |h| |f 0 (c2)| ≤ 2M |h| for some cj 's and all h ∈ R. Hence P = cos y, Q = 0, R = xy, and (P, Q, R) ◦ φ = (cos(cos t/ 2), 0, cos2 t/2). 3.4.0 a) False. Thus bk (f) = 4/(kπ) when k is odd and 0 when k is even. b) Repeat the proof of Theorem 3.6, replacing the absolute value by the norm sign. 10.5.7. Suppose H is compact. Then U = E and V = X \ E are nonempty open sets, U ∩ V = ∅, and X = U ∪ V . Let ε > 0 and set δ = ε/M . Hence, 2 |ak | ≤ |ak | for large k, and it follows from the Comparison Theorem that k=1 ak converges absolutely. And, if y = −1 then f (x, y) = x3 − 3x + 1 has critical points x = ±1 which correspond to extreme points of H. −2 40 Copyright © 2010 Pearson Education, Inc. c) The closure is R, the interior is R, the boundary is ∅. ∂u ∂v ∂v ∂u E 2 G2 − F 2 = 142 Copyright © 2010 Pearson Education, Inc. Thus f is absolutely integrable on (a, b) by the Comparison Theorem. 13.5.9. a) By hypothesis, F = (fx , fy). g ◦ φ(t) kφ (t)k dt = τ (Jk0) ∪N τ (Jk0) k=1 Since τ is 1-1, τ (Jk) \ τ (Jk0) consists of two points, so this last integral is unchanged if Jk0 is replaced by Jk . b) If the Riemann sums converge to I(f), then there is a partition P such that |S(f, P, tj)| < |I(f)| + 1 for all choices of tj ∈ [xj−1 , xj]. 3xn−1 + 6xn−1 + 4 3x2n−1 + 6xn−1 + 4 Using an initial guess of x0 = 0, we obtain x1 = −.25, x2 = −.313953 . √ c) The inequality is equivalent to n2 + 1 > 500. By definition and Exercise 5.3.7d, L(w) = x + y = L(s) + L(t) = L(st). Then x < x − 1 so 0 < −1, i.e., this case is empty. b) By definition, a and b are parallel if and only if a = tb, i.e., |a · b| = |t| kak2 = kak kbk. Then A and B are closed, A ∩ B = ∅, but dist (A, B) = 0 because 1/x → 0 as x → ∞. f 0 (0) π (g −1)0 (2) = 1 1 = . b) By vector algebra and the Cauchy-Schwarz inequality, |x·y−x·z| = |x·(y−z)| ≤ kxk ky−zk < 2·(3+4) = 14. Conversely, suppose xn ∈ E \ {a} and xn → a as n → ∞. By Theorem 9.26, f −1 (U) ∩ E is relatively open in E. Therefore, the series converges by the Alternating Series Test. Thus such a vector has the form (a, (20 − 8a)/7, (8 + a)/7), a 6= 0. ˜∞ R∞ 5.4.2. a) If p 6= 1 then 1 dx/xp = x1−p /(1 − p) ˜1 . Thus f (x) ≤≤ M1 +· · ·+MN =: M for all x ∈ E ∗ ⊃ E. 9.1.8. a) If E ∩ Br (a) contains infinitely many points, then surely E ∩ Br (a) \ {a} is nonempty. d) We cannot multiply by the denominator x − 1 unless we consider its sign. b) This is the set of points on or inside the ellipse x2 + 4y 2 = 1. Since x ∈ [1, 3] implies |3 − x36 | ≤ 3 + 336 and x3 + nx66 ≥ 0 + n = n, it follows that ˜ ˜ ˜ nx99 + 3 ˜ |3 − x36 | 3 + 336 3 + 336 33 ˜ ˜ − x = ≤ ≤ 1 implies e4/N > 1 and e4/N → 1 as N → ∞, given ε > 0, we can choose N ∈ N so that 2 0 < e4/N − 1 < ε. 14.1.6. a) Since f (x) cos kx is odd, ak (f) = 0 for k = 0, 1, . c) By hypothesis, 0 < x1 < 1. SN f (x) ≥ 0+· · ·+fxj (x)+· · ·+0 = 1 for all x ∈ E. Hence we can use (−a, −b, c) for a normal at the point (a, b, c). Therefore, this series converges by the Alternating Series Test. Moreover, by the Monotone Property, sup A, sup B ≤ sup E. Then by the Approximation Property, there exist xj ∈ E such that kxj − ak → 0, i.e., such that xj → a. 10.2.6. Modify the proofs of Theorems 3.21 and 3.22, replacing the absolute value signs with the metric ρ. Thus Rx√= y + fx √ and we can set f = 0, P = 0, and g = 0. 2 2 Case 2. Fix x, y ∈ [xk−1 , xk]. If f is differentiable, then f is continuous on [a, b]. If α > 0 then |x|α → 0 as x → 0. 11.3.4. a) If (x, y) 6= (0, 0), then by the Chain Rule, a normal to K at (x, y, z) is given by (x/z, y/z, −1). Therefore, x is not a cluster point of [a, b). By (8) in 1.1, xn < ^{2} √ for n ≥ N , i.e., xn → 0 as n → ∞. Hence (ex − 1)/x = k=0 xk /(k + 1)! for x ∈ R. Thus given x1 < x2 , f (x1) = lim gn (x1) ≤ lim gn (x2) = f (x2). Conversely, suppose every relatively open covering of H has a finite subcover. If f (c) < 0, then c − a > 0 and c − b < 0 imply that f 0 (x1) < 0 < f 0 (x2). Then y = x and x = sin z. ∂q ∂x ∂q ∂y ∂q ∂z ∂q Hence by the Product Rule and Theorem 11.2, µ ¶ µ ¶ µ ¶ ∂2w ∂ ∂F ∂x ∂F ∂ 2 x ∂ ∂F ∂y ∂F ∂ 2 y ∂ ∂F ∂z ∂F ∂ 2 z = + + + + + 2 2 2 ∂p ∂p ∂x ∂p ∂x ∂p ∂p ∂y ∂p ∂p ∂z ∂p ∂z ∂p2 2 2 2 ∂F ∂ x ∂F ∂ y ∂F ∂ z = + + ∂x ∂p2 ∂y ∂p2 ∂z ∂p2 µ ¶ µ ¶2 µ ¶2 2 2 2 ∂F ∂x ∂ F ∂y ∂ 2 F ∂z + + ∂x2 ∂p ∂y 2 ∂p ∂z 2 ∂p µ ¶ µ ¶ µ ¶µ ¶ µ ¶µ ¶ 2 2 ∂ F ∂x ∂y ∂ F ∂x ∂z ∂ 2 F ∂y ∂z +2 +2 +2 . Thus the series converges absolutely on (−1/e, 1/e). √ √ d) Suppose 3 < a < 5. b) The ratio of successive terms of this series is 2k + 1 4 2 =1− =1− . We obtain x = 1 − 1 − x, i.e., x2 − x = 0. Since each Vα is nonempty, choose a point xα ∈ Vα for each α ∈ A. 2 2 2 Thus {xn } is decreasing and bounded below. 2.3.8. a) This follows immediately from Exercise 1.2.6. p √ b) By a), xn+1 = (xn + yn)/2 < 2xn /2 = xn . The trace looks like a gull in flight (called a cubical parabola) traced in the z = x plane. If n ≥ N := max{N1 , N2 } and x ∈ E then |(f g)(x) − (fn gn)(x)| ≤ |f (x) − fn (x)| |gn (x)| + |f (x)| |g(x) − gn (x)| 0 choose δ > 0 so small that x, y ∈ E and |x − y| < δ imply |fN (x) − fN (y)| < ^{2}/3. 91 Copyright © 2010 Pearson Education, Inc. 2 (a + b) sin2 ϕ + 0 13.5.4. a) By Gauss' Theorem, ZZ Z 2 Z 4 Z Z 1 ω= 2 (yz + 2y + 1) dz dy dx = 2 S x2 −2 0 (24 − x2 − 5x4 /4) dx = 224/3. Then by Exercise 1.2.5b, 0 < xn < xn+1 . Therefore, kf kH is finite and attained by the Extreme Value Theorem. Chapter 4 4.1 The Derivative. It follows from the Squeeze Theorem that f × g is differentiable and its total derivative is T . Then µ kψu k2 = ∂x ∂u ¶2 µ + ∂y ∂u ¶2 µ + and ψu · ψv = ∂z ∂u ¶2 µ , kψv k2 = ∂x ∂v ¶2 µ + ∂y ∂v ¶2 µ + ∂z ∂v ¶2 ∂x ∂x ∂y ∂y ∂z ∂z + + . e) Integrating by parts twice, Z ∞ Z Z b ∞ −st b b2 ∞ −st e−st sin bt dt = e cos bt dt = 2 − 2 e sin bt dt s 0 s s 0 0 R∞ for s > 0. Hence by Theorem 11.15, g is differentiable on V . Also, 0 ≤ s2n+1 − s2n = 1/(2n + 1) → 0 as n → ∞. 2k)/(1 · 3 . f) By Cauchy-Schwarz and Remark 8.10, |x · (y × z)| ≤ kxk kyk kzk < 1 · 2 · 3 = 6. (σ, J) runs counterclockwise. Therefore, f is not differentiable at (0, 0). Thus by induction, 0 < xn < xn+1 for all n ∈ N. Conversely, suppose x ∈ f −1 (f (A)). Then y = x2 and z = sin x. Let x ∈ E and khk < δ. dx j=1 n X (−1)j−1 xj dx1 . But x is negative on (−∞, −6], so we must use the negative sign. b) By construction, to pass from Ek−1 to Ek we eliminate each point which has a 1 as the kth digit in all of its ternary expansions. x − 1 > 0. dxn = n X (−1)j−1 Vol (E). Let h = αh0 , α 6= 0, where khk < δ and let c ∈ L(a; a + h). Indeed, 0 ≤ f 0 (x0) ≤ 0, i.e., f 0 (x0) = 0. c) Let ε > 0 and choose (by Archimedes) an N1 ∈ N such that k > N1 implies 1/k < ε/2. b) Clearly, kf k1 ≥ 0 and kf k1 = 0 if and only if f = 0 (see Exercise 5.1.4b). 2 Next, notice by part a) that f 0 (x) = (2/x3)e−1/x for x 6= 0 and f 0 (0) = 0. a) Suppose E is convex but not connected. 4x2 − 1 < 0. Hence D(f + g)(x, y) = [2x + 1 2y − 1] b) Df (x, y) = [y and D(f · g)(x, y) = [3x2 − 2xy + y 2 2xy − x2 − 3y 2] . Let ^{2} > 0 and choose a grid G = P × Q on R, where P = {a0 , . Since differentiability implies continuity, it follows from the Inverse Function Theorem that f −1 is differentiable on f (I), and (f −1)0 (x) = 1/f 0 (f −1 (x)). 8.4.2. a) This is the set of points on or inside the ellipse x2 + 4y 2 = 1. Then there is a δ0 > 0 such that 0 < c − x < δ0 implies F (c) − F (x) > ε0 /2. It follows that fxj (0) = 0 for j = 1, . Then f satisfies the condition for α = 1 but fx (0, 0) = limh→0 |h|/h does not exist. Hence it follows from Parseval's Identity that 4 ∞ X 1 π (a2k (f) + b2k (f)) sin2 kh = k=1 Z π |f (x + h) − f (x − h)|2 dx −π for each h ∈ R. Then 2 e−1/x 1/xj+1 (j + 1)/xj+2 (j + 1)/xj−1 = lim = lim = lim . 9.2.6. For each x ∈ K, f is constant on Bδx (x) ∩ K. Since L is the line segment between (a + c)/2 and (b + c)/2, L has direction v := a+c b+c a−b − = 2 2 2 and length kvk. 8.1.4. a) Let θ be the angle between φ(t1) − φ(t0) and φ(t2) − φ(t0). 12.5.3. If f ∈ Cc∞ (R) then f = 0 on some interval (a, b). It converges to the continuous function 0 as k → ∞, so by Dini's Theorem, (log k + x)/(k + x) → 0 uniformly on [0, 1] as k → ∞. Case 2. If it holds for n then n+1 X k=1 a−1 1 a−1 1 = 1 − n + n+1 = 1 − n+1 . 12.1.3. By Remark 12.6, every rectangle is a Jordan region. c) (k + 1)/k ≥ 1 so the terms of this series are all ≥ 1. So, let x ∈ E and choose xk ∈ D such that xk → x. Moreover, xn−1 + 1 − 1 ≤ 1 − 1 = 0. d) Repeat the proof of Theorem 2.29. By the Triangle Inequality, n ≥ N implies ρ(xn , yn) ≤ ρ(xn , a) + ρ(yn , a) < ε. By definition, f (A \ B) ⊇ f (A) \ f (B) holds whether f is 1–1 or not. Therefore, {sn } is bounded. 10.3.9. Suppose f : R → R is continuous and I = (a, b). By definition and Exercise 5.3.7c, L(y) = xq = qL(t) = L(tq). Thus (f −1)0 (xn) = 1/f 0 (f −1 (xn)) → 1/f 0 (f −1 (x0)) = (f −1)0 (x0) as n → ∞. n→∞ n→∞ n→∞ The remaining two inequalities follow from Exercise 2.5.2. For example, lim sup xn + lim inf yn = − lim inf (−xn) − lim sup(−yn) n→∞ n→∞ n→∞ n→∞ ≤ − lim inf (−xn − yn) = lim sup(xn + yn). Let w = F (x, y, f (x, y)). c) By repeating the proof of Theorem 2.8, we can show that every Cauchy sequence is bounded. xp+1 By Remark 5.46 and Exercise 5.4.2a, this last integral is absolutely integrable since p + 1 > 1. If w ∈ B^2 (y) then ρ(w, a) ≤ ρ(w, y) + ρ(y, a) < r − ρ(y, a) + ρ(y, a) = r and ρ(w, a) ≥ ρ(y, a) − ρ(w, y) > ρ(y, a) + s − ρ(y, a) = s. Thus {φj ψk } is a C p partition of unity on V subordinate to the covering {Vj ∩ Wk }. Since A and B are compact (hence bounded—see Theorem 10.46), use the Bolzano-Weierstrass Property to choose subsequences such that xkj → x0 ∈ A and ykj → y0 ∈ B. 11.3.8. a) dz = 2x dx + 2y dy. √ √ √ √ √ √ b) If n + 3 + n ∈ Q then + 3 + n)2 ∈ Q. 2 2 x→0 xj+1 x→0 e1/x x→0 2e1/x /x3 x→0 2e1/x2 L := lim If j − 1 ≤ 0 then 1/xj−1 = x1−j is bounded near 0, hence L = 0. Given ^{2} > 0 choose N0 so large that k ≥ N0 and x ∈ E imply |fk (x) − f (x)| < ^{2}/2. By Definition 13.6 and the Fundamental Theorem of Calculus, ds/dt = kφ0 (t)k, hence by the Inverse Function Theorem, dt/ds = (`−1 )0 (s) = 1/kφ0 (t)k, where t = `−1 (s). b) Following Example 2.13, n3 + n − 2 1 + (1/n2) − (2/n3) 1 = → 3 2n + n − 2 2 + (1/n2) − (2/n3) 2 as n → ∞. Therefore, 1/(2x2 + x − 3 ≥ M . d) The graph of x2 − 2x + 2 is a parabola whose minimum is 1 at x = 1. By Remark 10.44, A and B are closed sets, so by Theorem 10.31, A ∩ B is a closed subset of the compact set A. On the other hand, since w + 1 is not the supremum of E, w + 1 > 10n+1 y. and D(f · g)(x, y) = [xy cos x + 2xy sin x − y cos y c) Since · −y sin(xy) −x sin(xy) Df (x, y) = log y x/y we have · D(f + g)(x, y) = xy sin y + x2 sin x − x cos y] . The limit is continuous because φ0 is continuous. Let x ∈ [c, d]. This point is outside H so can be disregarded. However, the product of any irrational with 0 is a rational. In particular, it follows from Exercise 1.6.5c that f is 1–1 if and only if f is onto. Finally, by the Chain Rule, (xα)0 = (E(αL(x)))0 = E(αL(x)) · αL0 (x) = xα α = αxα−1 . Since xyz = 16, neither x nor y is zero. Thus f (x/m) = f (x)/m for m ∈ N. Hence by the Intermediate Value Theorem, there is an x (between −1 and 0) such that f (x) = 0. Hence x ∈ BX ∩ Y ⊂ V , i.e., V is open in Y . 0 0 Hence by the Fundamental Theorem of Calculus, Z y fx = Pv (x, v) dv + h0 (x) = P (x, y) − P (x, 0) + h0 (x). 1.3 The Completeness Axiom. 11.7.6. Suppose D(2) f (a)(h0) < 0 for some h0 ∈ R2 . Hence if a 6= 0, then it follows from the Root Test that this series converges absolutely when a|x| < 1, i.e., |x| < 1/a. 10.2.4. a) Surely a set which has infinitely many points is nonempty. Since gn → g and g is bounded by M , choose N2 so large that |gn (x)| ≤ 2M for all n ≥ N2 and x ∈ E. 108 Copyright © 2010 Pearson Education, Inc. C1 : x2 + z 2 = 8, y = 1, oriented in the clockwise direction when viewed from far out the y axis, and C2 : x2 + z 2 = 8, y = 0, oriented in the counterclockwise direction when viewed from far out the y axis. Let φ(x) = xe−x . Since f 0 (x) = −1/(1 + x2) < 0 for all x ∈ R, f (k) ↓ 0 as k → ∞. We conclude that f (b) = 0. Then dx = dy = 2t dt, dz = dt, and Z √ Z 0 x dx + cos y dy − dz = C (2t2 + 2t cos(t2) − 1) dt = 1 1 − sin(1). Thus choose |x − y| < δ and x, y ∈ E0 imply |f (x) − f (y)| 0, choose an integer n > 1 so large that 1/n < ^{2}/2 and choose δ > 0 so small that 2(n − 1)δ < ^{2}/2. Thus |f (x, y)| ≤ M |x − 1|, and it follows from the Squeeze Theorem that f (x, y) → 0 =: L as (x, y) → (1, b). By the Approximation Property, choose xk ∈ E such that xk → sup E. Since Faa = 2 k=1 x2k > 0, this critical point is either a minimum or a saddle point. Let δ > 0 be so small that Bδ (xj) ⊂ Rj0 and define A as above. 1.3.2. Since a − 1/n < a + 1/n, choose rn ∈ Q such that a − 1/n < rn < a + 1/n, i.e., |a − rn | < 1/n. b) Let ε > 0 and set δ := ε. This is sometimes called the Reciprocal Rule. Let f (x) = x and g(x) = 1 if x < 2 and 2 if x ≥ 2. c) By b), (1 + 1/n)n ≥ 1n + n1n−1 (1/n) = 2. Otherwise, given any M > 0 there is an N ∈ N such that xN > M . Hence its arc length is 2 2 2 4 + 4 + 4 = 4 3. x] and Dg(x, y) = [x cos x + sin x sin y]. Hence by Theorem 10.34, Ao ⊆ B o . If x ∈ A, then x ≤ sup A. Since each of the numbers x0 , y0 , u0 , v0 , s0 , t0 is nonzero, this determinant is nonzero. As t → ∞, (x, y) → (0, 0) and dy/dx = (2t − t4)/(1 − 2t3) → ∞. √ b) Let f (x) = x − log x − 0.6. Since x ≥ 4, √ √ x−2 x √ f 0 (x) = 1/(2 x) − 1/x = ≥ 0. Hence sn converges by Theorem 2.29. Moreover, by Exercise 12.5.1, products of functions with compact support have compact support. Since (2/3)k → 0 as k → ∞, it follows that E is of measure zero. By Gauss' Theorem, ZZ ZZZ Z Z 1 Z Z 1 y|z| dV = 2 yz dy d(x, z) + 2 yz dy d(x, z) √ A 0 B x2 +z 2 −1 Z H Z = z d(x, z) + (2 − x2 − z 2)z d(x, z) ω= S Z A π Z = B 1 Z 2 π √ 2 Z r sin θ dr dθ + 0 0 0 √ (2r2 − r4) sin θ dr dθ = 2(8 2 − 2)/15. 20 Copyright © 2010 Pearson Education, Inc. Combine (6) and (7). Let (x0 , y0) be a point where x0 = y0 . Since f 2 ≥ 0, we conclude that f 2 , hence f , is identically zero on [a, b]. Therefore, Vol (x + E) := inf G X |x + Rj | = inf G Rj ∩E6=∅ X |Rj | = Vol (E) Rj ∩E6=∅ b) Since φ(x) := αx is C 1 and ∆φ = αn 6= 0, it is clear by Theorem 12.10 that αE is a Jordan region if and only if E is. c) By part b) and the Extreme Value Theorem, 1/|g(x0)| = inf x∈E 1/g(x) is positive. R Conversely, by part b) the function f (x, y) := C(x,y) F · T ds is well-defined. 4.5 Inverse Functions. Therefore, 0 < a < 2 + a − √2 = b. Thus y ∈ Br (a) ∩ Bs (b). Then f is C ∞ on R. (Indeed, in either case, q < p so this expression is eventually bigger P than q.) ∞ The inequality implies |ak+1 /ak | < 1 − q/k for k large. By Theorem

9.6, it follows that fk → some function f pointwise on H. Hence n−1 U (f, P) = 2 X 2 + 2δ = + 2(n − 1)δ < ²/2 + ²/2 = ². f (x) > |2M |/2 = M . 14.5.5. Suppose F is not convex. Since E is bounded above (by a), it follows from the Completeness Axiom and Theorem 1.15 that n0 = sup E exists and belongs to E. Thus x=− a , 2cD y=− b , 2cE and z= 1 4c2 μ a2 b2 + D E ¶ .) onto the number with decimal expansion 0.x1 x2 · · · . But this follows immediately from the Squeeze Theorem since φ(N) is bounded as N → ∞ and e−(s−a)N → 0 as N → ∞ for any s ≥ b > a. 7.3.10. Since ak ≥ 0, it follows that sn → ∞ as n → ∞. In particular, all the functions which appear in this problem are products of integrable and/or continuous functions, hence integrable. In particular, X contains more than two clopen sets. b) Suppose f is increasing and continuous on (a, b). kxk kyk n(b − a)2 76 Copyright © 2010 Pearson Education, Inc. Telescoping, we obtain Z F · T ds = C N X f ◦ φj (bj) − f ◦ φj (aj) = f ◦ φN (bN) − f ◦ φ1 (a1) = 0 j=1 since C is closed. Now for k > e, log(1/|ak |) log(log k log log k) (log log k)2 log2 u = = ≡ . Suppose it holds for some n. Finally, letting r0 ↓ r, we √ k conclude that lim supk→∞ ak ≤ r, as required. d) Let η > 0 and let C² be the constants in part c). Thus choose M > 0 such that |xn | and |yn | are both ≤ M for all n ∈ N. Consequently, if j > J then (since k + j > k) ¯N ¯ ∞ ∞ ¯X a ¯ X X ak |ak | ¯ k ¯ ≤¯ + < ². Thus g is well defined on all of X. 11.5 The Mean Value Theorem and Taylor's Formula. Therefore, sk 1 > y and x + br1 ≤ s` ≤ y + bk1 for all k1 ≤ ` ≤ r1 . k≥n k≥n Taking the infimum of this inequality over all j ≥ n, we obtain inf k≥n (xk +yk) ≤ supk≥n xk +inf j≥n yj . nn nn ≤ n! en−1 ≤ nn+1 for all n ∈ N. But by part a), Ac ∪(∂B)c ⊆ ∂A and B c ∪(∂A)c ⊆ ∂B. Hence by hypothesis, there is an n ∈ N and a 1–1 function φ from E onto {1, 2, . 13.1.5. a) This curve evidently lies on the cone x2 + y 2 = z 2 . b) Let F = (x(x2 + y 2 + z 2)−3/2 , y(x2 + y 2 + z 2)−3/2 , z(x2 + y 2 + z 2)−3/2) and E = B1 (0, 0, 0). 1.2.8. a) Since (1 − n)/(1 − n2) = 1/(1 + n), the inequality is equivalent to 1/(n + 1) < .01 = 1/100. Then x ∈ Bδxj (xj) for some 1 ≤ j ≤ N , so f (x) = f (xj). It is closed and connected. Therefore, by Dini's Theorem and Theorem 7.10, Z 1 lim k→∞ μ x2 f 0 k k2 + x ¶ Z 1 dx = x2 f (0) dx = 0 f (0) . If y = 0, 1 ≤ x ≤ 3, then f (x, y) = x2 takes its minimum at x = 0. To compute the value of A(x) when x = p0 ∈ Q. Thus n > 39/2, i.e., n ≥ 20. Moreover, since fxk is continuous on E ∗ := k=1 [xk −rk , xk +rk], the Extreme Value Theorem implies that there are constants Mk that |fxk | ≤ Mk on E ∗ for all k. gx (a, b, c) gy (a, b, c) gz (a, b, c) In particular, fx (a, b, c)gz (a, b, c) − fz (a, b, c)gx (a, b, c) = 0 = fy (a, b, c)gz (a, b, c) − fz (a, b, c)gy (a, b, c). 0 On the other hand, ∂T has three pieces: C1 which runs from (1, 0, 0) to (0, 0, 1), C2 which runs from (0, 0, 1) to (0, 1, 0), and C3 which runs from (0, 1, 0) to (1, 0, 0). Define g on E by g = f . Hence by d), {y} = f ({a}) ∩ f ({b}) = ∅, a contradiction. On the other hand, if x ∈ E then since E o = ∅, Br (x) is not contained in E for any r > 0, i.e., Br (x) ∩ E c 6= ∅ for all r > 0. Thus n ≥ N implies |1 + 2xn − 3| ≡ 2 |xn − 1| < ε. It converges to the continuous function sin x 1/2 as k → ∞. Hence differentiating term by term, we obtain ¯∞ ¯ ∞ ¯X cos(x/(k + 1)) ¯ X 1 ¯ ¯ 0 |f (x)| = ¯ = 1. 3 6 √ 1.4.9. a) If√m = k 2 , then m = k by definition. Hence x = z/2 and y = z/2. Suppose xn > 2. It also diverges for p ≤ 0 by the Divergence Test. 10.1.5. a) Let a be the common limit point. c) Consider f (x) = 2x + 3x − 2. Since f 0 is continuous, it follows from the sign preserving lemma that there is an interval I ⊂ (a, b) containing x0 such that f 0 (x) > 0 for all x ∈ I. b) By Exercises 11.1.8 and 11.1.9, L{t sin πt} = −L{sin πt}0 (s) = −(π/(s2 + π 2))0 = 2sπ/(s2 + π 2)2 . If f 0 < 0 on [a, b], then set ψ(u) = (f −1 (−u), −u) for u ∈ [−f (a), −f (b)] and τ (u) = f −1 (−u). Thus xn ∈ Br (a) ∩ E. But by (2), cos θ = (a × b) · c/(ka × bk kck). Hence (1, 1, 1) and (−1, −1, −1) are the only points where the tangent plane of H is parallel to x + y − z = 1. This function is continuous, and f (−1) = 1/e − 1 < 0 < 1 = f (0). Thus the point on the paraboloid where the tangent plane is parallel to x + y + z = 1 is (−1/2, −1/2, 1/2) and an equation of this tangent plane is 2x + 2y + 2z = −1. By elementary set algebra and Theorem 1.37, (f −1 (A) ∩ f −1 (B)) ∪ f −1 (C) = f −1 (A ∩ B) ∪ f −1 (C) ⊃ f −1 (A ∩ B) 6= ∅. sin t c) By part b) and a change of variables, (*) π 2 (S2N f)( )= 2N π Z π/2N 0 sin 2N t 2 dt = sin t π Z 0 π sin u du. c) Since √ √ p (k − k 2 + k)(k + k 2 + k) −k 1 √ √ k − k2 + k = = →− 2 k + k2 + k k + k2 + k √ as k → ∞ and by l'Hˆopital's Rule, k 1/k → e0 = 1 as k → ∞, we see by Theorem 9.2 that (k− k 2 − k, k 1/k , 1/k) → (−1/2, 1, 0) as k → ∞. c) If f is not increasing on (a, b) then there are points c < d in (a, b) such that f (d) < f (c). By symmetry, then, 2.9253226 = 2 · (1.4626613) < −1 ex dx < 2.9253626. Since H is compact, there exists a finite subset A0 of A such that {Vα }α∈A0 covers H. 11.3.10. Since it is nonnegative on ∂H, there is a point (x1 , t1) ∈ H 0 such that u(x1 , t1) < 0. By the Triangle Inequality, n ≥ N implies |xn − yn | ≤ |xn − a| + |yn − a| < ε. f (x) ≥ g(x). , x3 = −.31766 . If −δ < x < 0, then f (x) = −x/x = −1 so |f (x) − L| = | − 1 + 1| = √ 0 < ε. n→∞ n→∞ n→∞ c) Let xn = (−1)n and yn = (−1)n+1 . By Lebesgue's Theorem, f is almost everywhere continuous. If x = −1 then f (x, y) = −1 − 3y − y 3 which has no critical points. d) Let (x, y, z) = φ(t). , JN such that τ 0 > 0 on each Jk0 and J = ∪N k=1 Jk . If P (x) = an xn + · · · + a0 , then it follows from Theorem 4.10 that P 0 (x) = nan xn−1 + · · · + a1 exists and is a polynomial. If a + b + c = 4 then b = (20 − 8a)/7, c = (8 + a)/7. 14.1.1. a) Since x2 sin kx is odd, bk (x2) = 0 for k = 1, 2, . Hence by Exercise 3.18, f ∨ g and f ∧ g are absolutely integrable on (a, b). Since F (0, 0, 0) = 0 and ∂F 3 = 2z + p 2 ∂z 2 sin(x + y 2) + 3z + 4 equals 3/4 6= 0 at (0, 0, 0), the expression has a differentiable solution near (0, 0, 0) by the Implicit Function Theorem. In particular, 1/xn < 2. Finally, since P0 is finer than P , U (f, P0) ≤ U (f, P). By the argument in part b), (2, xn) are closed and bounded, √ but not compact. 8.3.10. The result is false if "open" is omitted. b) Let V be relatively open in f (E), i.e., V = U ∩ f (E) for some U open in Y . Thus the Taylor series contains only odd terms. 9.4.1. a) f (0, π) = (0, 1] is not open and we don't expect it to be; f [0, π] = [0, 1] is compact and connected as Theorems 9.29 and 9.30 say it should; f (−1, 1) = (− sin 1, sin 1) is open, big deal; f [−1, 1] = [− sin 1, sin 1] is compact and connected as Theorems 9.29 and 9.30 say it should. Then p √ √ √ F · φ0 = (−1 + cos3 t + 5, sin t/ 2, 1) · (0, − sin t, cos t/ 2) = (cos t − sin2 t)/ 2. Thus limx→0 log x does not exist. b) By parts and part a), Z ∞ Z Z n ∞ n−1 −st n! ∞ −st n! tn e−st dt = t e dt = · · · = n e dt = n+1 s 0 s 0 s 0 for s R> 0. Then `(t) = r(t − t0), and θ(t) = t − t0 . Hence by the Comparison Theorem, f is not improperly integrable on [a, b). g) Clearly xn → ∞ as n → ∞. Since ∇fy (a, b) · (h, th) − ∇fy (a, b) · (0, th) = hfyx (a, b) we can write ∆(h) = fy (a + h, b + th) − fy (a, b + th) h = fy (a + h, b + th) − fy (a, b) − ∇fy (a, b) · (h, th) − (fy (a, b + th) − fy (a, b) − ∇fy (a, b) · (0, th)) + hfyx (a, b). 5.4.3. Since p > 0, integration by parts yields Z 1 ∞ sin x cos x ¯∞ dx = − p ¯1 −p xp x Z 1 ∞ cos x dx = cos(1) − p xp+1 Z 1 ∞ cos x dx. Since y = x2 + 2x − 5 is a quadratic in x, we √ have x = (−2 ± 4 + 4(5 +√y))/2 = −1 ± 6 + y. If x0 < x < x0 + δ then P (x)/(x − x0) > m0 /δ ≥ m0 (M/m0) = M . 4.2.1. a) By the Product Rule, (f g)0 (2) = f 0 (2)g(2) + f (2)g 0 (2) = 3a + c. Hence f −1 (V) ∩ E is relatively open in E. By Exercise 12.2.3, we have E ZZ 1 ∆u(x0) = lim 2 ∆u dA = 0. d) Applying L'Hˆopital's Rule twice, we obtain √ √ √ √ k log(k/(k − 1)) log(k/(k − 1)) = lim k→∞ k→∞ log k log k log k/k √ √ k−1 −k 2 /2 k = lim √ lim √ k→∞ k k→∞ (k − 1)2 (1 − log k) √ k − k/2 √ = lim lim k→∞ (k − 1)2 k→∞ 1 − log k √ −1/(4 k) = lim = ∞. Since kNφ k = k(f 0 (u), 0, such that U (f, P) − L(f, P) < 2 c ε. 14.2.6. Let Z 2π ∆N (x) = f (x − t)φN (t) dt − f (x). Since 2k p ≤ k=1 1/ k √ + 1 converges if and √ √ k 2p + 1 + k p ≤ 2 k 2p + 1 implies 1/2(2 k 2p + 1) ≤ 1/(k 2p + 1 + k p) ≤ 1/(2k p), it follows from the Comparison Test that the original series converges if and only if p > 1. Thus k=1 |Ik | ≥ 1. Moreover, by the Binomial Formula, the inductive hypothesis, and what we just proved, a2n−1 + b2n−1 a2n + b2n = (an−1 + 1)2 + (2an−1 + bn−1 + 1)2 = a2n−1 + 4an−1 + 4 + (2an−1 + 1)2 + 2bn−1 (2an−1 + 1) + b2n−1 = c2n−1 + 2(an−1 + 2) + (2an−1 + 1)2 + 2(cn−1 − 1)(2an−1 + 2) = c2n−1 + (2an−1 + 2)2 + 2cn−1 (2an−1 + 2) = (2an−1 + cn−1 + 2)2 ≡ c2n . Thus f (C \ B) = {0} 6= ∅ = f (C) \ f (B). By using the change of variables u = x + h, du = dx, and a sum angle formula, we have Z 1 π ak (f (x + h)) = f (x + h) cos kx dx π −π Z π 1 = f (u)(cos ku cos kh + sin ku sin kh) du π −π = ak (f) cos kh + bk (f) sin kh. In particular, (P, Q) ◦ φ = (36, sin t, cos2 t). By part a), |ax0 − ax | = ax0 − ax < ar − aq = aq · (ar−q − 1) < ax0 (a1/N − 1) < ε. By hypothesis, g and f are differentiable on I \ {a}, the limit of g/f is of the form 0/0 or ∞/∞, and the limit of g 0 /f 0 is zero. Since q < x and x < y imply q < y, it is clear by definition that ax ≤ ay . 4.4.3. By Taylor's Formula, there is a c between x and 0 such that µ ¶ xn ec xn+1 ex = 1 + x + · · · + = . By part b), given any x ∈ E the set Ux which can be polygonally connected to x through E is open. d) Since · 0 1 Df (x, y, z) = 1 0 0 −1 ¸ · and Dg(x, y, z) = yz 1 xz 2y ¸ xy , 0 we have · D(f + g)(x, y, z) = yz + 1 xz + 1 2y xy −1 ¸ , and D(f · g)(x, y, z) = [y 2 (z + 1) 2xyz + 2xy − 2yz y 2 (x − 1)] . Then f satisfies the hypotheses with α = 1 and I = [−1, 1], but f is not differentiable at 0 ∈ [−1, 1]. b) 0 = fx = 2x + 2y and 0 = fy = 2x + 6y imply x = y = 0. By Example 2.21, |1|1/n and (b − a)1/n → 1 as n → ∞. √ √ √ √ 1.2.6. a + b − 2 ab = (a − b)2 ≥ 0 for all a, b ∈ [0, ∞). Choose N1 so large that n ≥ N1 and x ∈ E imply |fn (x) − f (x)| < ²/(3M) and |gn (x) − g(x)| < ²/(3M). Choose s so small that N sn < ². If n ≥ N and x ∈ [a, b] then ½ |fn (x)g(x) − f (x)g(x)| = |fn (x) − f (x)| |g(x)| 0 choose N so large that n ≥ N and x ∈ E imply |fn (x) − f (x)| < ²/ max{2, |α| + 1} and |gn (x) − g(x)| < ²/ max{2, |α| + 1}. If x2k = log k and xn = 0 for n 6= 2k , then x2k − x2k−1 = log(k/(k − 1)) → 0 as k → ∞, but xk does not converge, hence cannot be Cauchy by Theorem 2.29. 0 Since the integral of φN is 1, notice that Z 2π ∆N (x) = (f (x − t) − f (x))φN (t) dt. Since fn → f uniformly and n1/n → 1, as n → ∞, choose an N so large that n ≥ N implies |fn (x) − f (x)| 1/r for some r ∈ R. If distinct points x1 , . Thus the directional derivatives of f exist. Then g is periodic and of bounded variation on [−π, π] and continuous on any interval [a, b] ⊂ (−π, π). Therefore, (f (h, k) − f (0, 0) − ∇f (0, 0) · (h, k))/k(h, k)k → 0 as (h, k) → (0, 0). If Qk is a cube of side s which contains xk then N X |Qk | = N sn < ². d) Since (x/3x)0 = (3x · 1 − x log 3 · 3x)/32x = (1 − x log 3)/3x < 0 for x ≥ 1, k/3k is decreasing. Then n ≥ N implies |(f ·fn)(x)−f 2 (x)| = |f (x)| |fn (x)−f (x)| < ε for x ∈ [a, b], i.e., f · fn → f 2 uniformly on [a, b] as n → ∞. By the Archimedean Principle, E is nonempty. 2 Therefore, |xn+1 − xn | = |f (xn)/f (xn)| ≤ M |xn − xn−1 |P /²0 < r02n+1 < r0n+2 . Suppose x1 < x2 < · · · < xn in E have been chosen so that sup E − 1/n < xn < sup E. Then Ac := Y \ A is open in Y , so by Corollary 10.59, A0 := f −1 (Ac) ∩ E is relatively open in E. Let ε > 0. d) Let x = t2 , y = t2 , z = t, and let t run from 1 to 0. 4 d) Set δ = x/k. Since c = 0 implies a = b = 0 and K has no tangent plane at the origin, there are no tangent planes to this cone which are parallel to x − y + z = 5. d) By Theorem 1.18, the closure of Q is R and the interior of Q is the empty set, so the boundary of Q is R. c) Let f (x) = 2|x| − sin2 x and suppose first that x ≥ 0. Thus f /g is locally integrable and |f |/|g| ≤ h/ε holds everywhere on (a, b). If u satisfies the heat equation, then wxx − wt = −r < 0 on V . But by √ (−1, 1) with (arcsin x) = 1/ 0 2 2 trigonometry, cos y = 1 − x . h→0 ρh ρ h→0 h fxj (ρx) = lim Thus by the Chain Rule, homogeneity, and the Power Rule, ρk−1 n X j=1 xj fxj (ρx) = n X j=1 xj fxj (ρx) = d d k (f (ρx)) = (ρ f (x)) = kpk−1 f (x). The sets Am := { nk : k ∈ N and − 2m ≤ k ≤ 2m } are finite, hence at most countable. Fix x ∈ [a, b]. If E is sequentially compact, then by part b), E is closed and bounded. Let f (x) = xα /ex . |t1 − t0 | |t2 − t0 | kbk2 Thus θ is 0 or π. Hence by Jensen's inequality, ¶¶1/q µZ |f (x)|p dx ≤ 0 1 ¶1/q φ(|f (x)|p) dx 0 µZ = 1 ¶1/q |f (x)|q dx . On the other hand, given x ∈ E, there is a sequence xj ∈ Br such that xj → x as j → ∞. 2 2 2 Thus {xn } is increasing and bounded above, so xn → x. 11.5.5. Let F be defined as in the proof of Theorem 11.35. Set f (y) = j=1 fxj (y). Let xn → ∞ and set yn = xn + δ/2. If 0 < x1 < x2 then Z x2 dt x2 − x1 L(x2) − L(x1) = > 0. Parameterize C1 by φ1 (t) = (0, −t, 1 + 2t), I1 = [−1/2, 0]; C2 by φ2 (t) = (t, 0, 1 − t), I2 = [0, 1]; and C3 by φ3 (t) = (−t, (1 + t)/2, 0), I3 = [−1, 0]. Taking the limit of xn−1 + 1 − 1 = xn we see that a2 + a = 0, i.e., a = −1, 0. 12.2.2. If x ∈ [0, 1]×· · ·×[0, 1], then 0 ≤ x2j ≤ 1. d) DR g(x) = DR f (x) + 1/n > DR f (x) ≥ 0 for all but countably many x ∈ (a, b). Thus (x, y) ∈ R and it follows from Theorem 10.16 that R is closed. 2 Therefore, −2 |x + 1| dx = 1/2(1) + 1/2(9) = 5. We check the boundary in four pieces. b) Suppose ∂E = E. 11.3.2. a) Since ∇f = (2x, 2y) = (2, −2) at (1, −1), and the equation of the tangent plane is z = f (1, −1) + ∇f (1, −1) · (x − 1, y + 1), we have z = 2x − 2y − 2. 1.3.8. Since A ⊆ E, any upper bound of E is an upper bound of A. b) All we need do is make the relative closed balls Ej in part a) nonoverlapping. Indeed, choose a, b such that K ⊆ [a, b] ⊂ (0, 2π) and define fe on (0, 2π) \ [a, b] to be linear, i.e., its 157 Copyright © 2010 Pearson Education, Inc. Thus n = (a, b, 0) and an equation of the tangent plane is ax + by = a2 + b2 = 1. 4.4 Taylor's Theorem and l'Hˆ opital's Rule. Notice that g(U ∩ V) = h(U ∩ V) = {u : u = (x2 , . xn xn 1+1 2 xn (1 + 1 − xn) 2.3.3. Case 1. By part a) and the law of cosines, θs kν 0 (s) − ν 0 (s0)k2 = 2 − 2 cos θs = 4 sin2 . We conclude √ √ that f (±2/ 5, ±1/ 5) = 0 is the minimum and f (±1/ 5, ∓2/ 5) = 5 is the maximum. Thus ∪k∈N [−1/k, 0] = [−1, 0]. Thus a ∈ f −1 (E) ∩ B. Thus by Stokes's Theorem ZZ Z curl F · n dσ = F · T ds = −1/12. b) Let fn (x) = 1/n and gn (x) = x. b) By the definition of the operator norm, ° ° ° Df (a)(h) ° kDf (a)(h)k kDf (a)k khkk ° ° ≤ = kDf (a)k ° khk ° = khk khk for all h 6= 0. Let f (a) = g(a) = 0 and x ∈ I \ {a}. This function is continuous, and f (0) = −2 < 0 < e + 1 = f (1). Differentiating term by term, we obtain E 0 (x) = solves the initial value problem y 0 − y = 0, y(0) = 1. Then U, V are nonempty open sets, U ∩ V = ∅ and U ∪ V = E. parametrization of the surface φx (Ex). Thus E is sequentially compact by definition. 8.2.3. All we have to do is find two lines which lie in parallel planes. Then 3(n + 1)2 + 3(n + 1) + 1 = 3n2 + 3n + 1 + 6n + 6 ≤ 2 · 3n + 6(n + 1). Taking the limit of this inequality as ² → 0, we obtain (U) a g(x) dx ≤ Rb Rb Rb f (x) dx. P∞Thus it follows that Sf is uniformly P∞ Abel summable to f . 0 b) This boundary is a triangle (oriented in the clockwise direction when viewed from the origin) consisting of three line segments, C1 (which lies in the yz plane), C2 (which lies in the xz plane), C3 (which lies in the xy plane). c) By Fej´er's Theorem, σN f → f uniformly. Hence, 0 < 1 − x < 2 and x2 − 1 < 0. |x1 − x0| = ¯¯ 0 f (x0) ¯ ²0 74 Copyright © 2010 Pearson Education, Inc. On the other hand, since f (a) is a local minimum we see by Taylor's Formula that 0 ≤ f (a + h) − f (a) = D(2) f (c)(h) () for c ∈ L(a; a + h) and khk sufficiently small. Hence it follows from Dini's Theorem that (1 − x/k)k → e−x as k → ∞ uniformly on any compact subset of R. 4.3.8. By the Mean Value Theorem, 0 < f (x1) − f (x2) = (x1 − x2)f 0 (c1) for some x1 < c1 < x2 and 0 < f (x3) − f (x2) = (x3 − x2)f 0 (c2) for some x2 < c2 < x3 . 0−∞ = ∞ is not indeterminate. Hence by Theorem 2.29, xn is Cauchy. 90 Copyright © 2010 Pearson Education, Inc. Hence 1/M < 1/|g(x)| < 1/²0 for all x ∈ [a, b] and it follows that 1/g is bounded on [a, b] and 1/(2M) < 1/|gn (x)| < ²0 for large n and all x ∈ [a, b], i.e., 1/gn is defined and bounded on [a, b]. Thus by the Monotone Convergence Theorem, xn → x for some x ∈ R. b) If E = α∈A Eα is empty or contains a single point, then E is connected by definition. Define f and g on [0, 1] by: f (x) = 1 when x ∈ Q and f (x) = 0 when x ∈ / Q, and g(x) = 1 + (x)−1 . Also notice that n X |gk (x)|2 = k=1 Hence for each j, hj (x) = x22 + · · · + x2n a2 − x21 a − x1 = = . In particular, Z y Z x f (x, y) = Q(x, v) dv + P (u, 0) du. Then (y − ², y + ²) ∩ E ⊂ {x1 , . Thus (V, h) is a chart in A which satisfies h(V) = B1 (0) and h(x) = 0. Hence ¯ n ¯ n−1 ¯X ¯ X ¯ ¯ ak bk ¯ ≤ 2M |an | + 2M |ak+1 − ak | . , ρ(xk , a)}, then by hypothesis there is an xk+1 ∈ E ∩ Bs (a). 0 Similarly, Z 0 Z C2 Z 1 P dx + Q dy + R dz = − and Z Q(0, y, 1 − y) dy 0 Z 1 P dx + Q dy + R dz = − C2 1 R(0, 1 − z, z) dz + 0 Z 1 P (x, 0, 1 − x) dx + Q(1 − y, y, 0) dy + 0 1 P (x, 1 − x, 0) dx. If ak = 1/k 2 and bk = 1/k, then ak /bk → 0 as k → ∞ and k=1 ak converges, but k=1 bk does not. d) The closure is R, the interior is ∅, the boundary is R. Therefore, Sf = Sg, i.e., S is the Fourier series of f . π 4k 2 − 1 k=1 P2n −1 Hence it follows from parts a) and b) that k=1 |φ(xk) − φ(xk−1)| > 2 log(2n)/π for all n ∈ N. If we set ta + x0 = ua + x0 implies t = u, C is simple. √ 7.3.9. The coefficients of this power series are given by ak = ((−1)k + 4)k − 4k . ak = 1/k is strictly decreasing to 0 but k=1 1/k diverges. By the Product Rule and the Fundamental Theorem of Calculus, Z b Z b (f 0 (x)g(x) + f (x)g 0 (x)) dx = a (f (x)g(x))0 , dx = f (b)g(b) − f (a)g(a). Therefore, ¯Z ¯ ¯ M |r|3 M |r|3 ¯¯ 2π 3 ¯≤ |R| ≤ (cos θ + sin θ) cos(2θ) dθ 16π, ¯ ¯ 6 6 0 i.e., |R|/r2 → 0 as r → 0. b) Suppose E is clopen and ∅ ⊂ E ⊂ X. (k + 1)log k (k + 1)5 k b) It converges by the Ratio Test, since (k + 1)100 /ek+1 ((k + 1)/k)100 1 = → k 100 /ek e e as k → ∞. k=0 P∞ If k=0 Sk rk converges, then SN rN → 0 as N → ∞, and we have verified the first identity. c) Since p > 1, choose α > 0 such that p − α > 1. n→∞ n→∞ n→∞ To obtain the reverse inequality, notice by the Approximation Property that for each n ∈ N there is a jn > n such that inf k≥n (xk + yk) > xjn − 1/n + yjn . The author and publisher shall not be liable in any event for incidental or consequential damages in connection with, or arising out of, the furnishing, performance, or use of these programs. Thus ψ(u) ≥ 0 for all u ∈ (−∞, 1), i.e., φ0 (t) ≥ 0 for all t > x. By the argument of Theorem 3.40, this definition is independent of the sequence xk chosen to approximate x. ⌉∂g1 /∂x1 (a) . Since HN ⊇ H1 , and HN ⊇ H1 , it follows that c ∅ = HN ∩ HN ⊇ H1 , which contradicts the that H1 is nonempty. Thus by part a), a cannot be a cluster point of R. 5.4.7. a) Suppose L > 0.R If f (x) → LRas x → ∞, then choose N ∈ N such that f (x) > L/2 for x ≥ N . b) Suppose that x < y. Hence ey/k cos(y/k) → 1 uniformly on E as k → ∞. Thus D2 F (x0) ≥ 0. We claim that L = 0. Thus by Taylor's Formula, √x+ √ y =3+ x − 1 y − 4 (x − 1)2 (y − 4)2 (x − 1)3 (y − 4)3 √ √ + − − + + 2 4 8 64 16 c5 16 d5 for some (c, d) ∈ L((x, y); (1, 4)). b) If E is a bounded infinite set, then it contains distinct points x1 , x2 , . Since f (x) exists and is nonzero for all x ∈ (0, ∞), it follows from Theorem 4.33 that f −1 is differentiable on (0, ∞) and (f −1)0 (x) = 1/f 0 (f −1 (x)). It is closed because its complement {(x, y) : 2x + 4y 2 > 1} is open. Iterating what we just proved, using the fact that the limit of the product is the product of the limits, we see that g(x) → f1 (a1) · · · fn (an) as x → a. a) Let s = `(t). Then Z Z Z Z (x + y + z 3) ds = (x + y + z 3) ds + (x + y + z 3) ds + (x + y + z 3) ds C C1 C2 C3 √ √ Z 1 Z 2 √ Z 1 1 3 5 5+3 5 = t dt + 5 (2 − t) dt + t dt = + +2= . The converse follows similarly integrating term by term. c) Since (−1)n+1 + (−1)n /n = −1 + 1/n when n is even and 1 − 1/n when n is odd, lim supn→∞ xn = 1 and lim inf n→∞ xn = −1. If x = 1 and z = 0, then 3 + y = 1 and −1 + w = 0, i.e., y = −2, w = 1. Given ² > 0 choose δ > 0 so small that a < x < a + δ or 65 Copyright © 2010 Pearson Education, Inc. Thus Z 2π Z π/2 A(S) = 0 a2 | cos v| dv du = 4πa2 . Now P (x) − P (y) = (x − y)Q(x, y) where Q(x, y) = an (xn−1 + · · · + y n−1) + · · · + a2 (x + y) + a1. When p = 1, the integral is log x ¯0 which diverges. If c = max{x−², a} and d = min{x+², b} then c < d and (x−², x+²)∩[a, b] ⊇ (c, d). The inequality holds if 4k 2 > 4k 2 − 1, i.e., if 1 > 0. c) Let ak = (2 · 4 . 5.2.0. a) True. Thus M is not an upper bound for one of the sets A or B, a contradiction. ak a a a d) The formula holds for n = 1. If one of the pair x, y belongs to [0, N] and the other does not, for example, if x ∈ [0, N] and y ∈ / [0, N], then |x − N | ≤ |x − y| < δ. C1 On C2 , x = 2 hence dx = 0, and 0 Z Z 3 y dx + x dy = 2 dy = 4. Thus by the calculations in part b), choose n = 11. , 8 but is a perfect square, namely 144 = 122 , when n = 9. Then mj (|f |) ≥ ² and it follows that Z b (L) |f (x)| dx ≥ L(|f |, P) ≥ mj (|f |)(xj − xj−1) ≥ ²(2δ) > 0. And, if 1 < p ≤ 2, then the maximum is 1 and the minimum is n(p−2)/p . Thus φ(t) is increasing for t > x, so eφ(k) = (1 − x/k)k ↑ e−x as k → ∞ for all x ∈ R. j=1 Thus f (H) ⊂ SN j=1 Vαj , i.e., f (H) is compact. By induction, there are infinitely many points in E ∩ Br (a). Thus k log k ≥ k p for k ≥ 3, and it follows from ∞ the Comparison Test that k=1 k − log k converges. ε) By definition, ⌠ x≤0 | ⌠ 3x + 2 f (x) = x+2 0 2. If k = 0 then nq = 1 is a root of the polynomial x − 1. , N }, ψ and τ are C 1 and τ 0 > 0. other hand, if fxy (a, b) = 0 then either fxx (a, b) 6= 0 or fyy (a, b) 6= 0. By definition, dT = 2π 1 2 r g L µ g dL − L dg g2 ¶ = T 2 µ dL dg − L g ¶ . But 0 is analytic on (−∞, ∞), so by Theorem 7.56, f = 0 on R. No part of this publication may be reproduced, stored in a retrieval system, or transmitted, in any form or by any means, electronic, mechanical, photocopying, recording, or otherwise, without the prior written permission of the publisher. If C(x, y) ends in a vertical line segment, then by part a), fy = Q. Thus xn+1 = 2xn yn < xn . Since |(1/α)Rj | = (1/α)n |Rj |, it follows that X X Vol (αE) = inf |αRj | = αn sup |Rj | = αn Vol (E). Hence x ∈ ∂E. Hence | k=1 (1 − cos(1/k))| ≤ 0 |f (x)| dx ≤ 2. The graph of f1 has no tangent at x = 0 because it oscillates between y = x and y = −x. In particular, {Uα }α∈A0 is a finite subcovering of E which covers H. Thus the original series diverges by the Divergence Test. is closed as Exercise 9.4.4 says it should; f −1 (−1, 1) = R \ {x : x = (2k + 1)π/2, k ∈ Z} is open as Theorem 9.26 says it should; f −1 [−1, 1] = R is closed as Exercise 9.4.4 says it should. 24m 24m Similarly, s(f ; Gm) = (24m − 23m+1 + 22m)/24m Consequently, S(f ; Gm) − s(f ; Gm) = 23m+2 /24m = 4/2m → 0 as m → ∞. By the Extreme Value Theorem, there is a t0 ∈ [a, c] such that f (t0) = F (c). Finally, since each fxj is increasing and nonconstant, there is a t ∈ E such that fx0 j (t) > 0 PN Thus f 0 (t) = k=1 fx0 k (t) ≥ fx0 j (t) > 0. In the same way, we can prove that a0 (f (x + h) − f (x − h)) = 0 and bk (f (x + h) − f (x − h)) = −2ak (f) sin kh for k ∈ N. Now n2 + 3n is a perfect square when n = 1 but if n > 1 then (n + 1)2 = n2 + 2n + 1 < n2 + 2n + n = n22 + 3n =< n2 + 4n + 4 = (n + 2)2 . When p = 1, the integral is log x ¯1 which diverges. Since k > sup E, k cannot belong to E, i.e., a < k. Since f (h, k) − f (0, 0) − ∇f (0, 0) · (h, k) h4 + k 4 = 2 ≤ 2(h2 + k 2)3/2−α → 0 k(h, k)k (h + k 2)α+1/2 as (h, k) → (0, 0), f is also differentiable at (0, 0). In particular, f (K) ⊂ {f (x1), . If x ∈ K, then x ∈ Ixj for some j, so f (x) ≥ 0 + · · · + fxj (x) + · · · + 0 > 0. Therefore V = ∪x∈V B² (x) as required. 4.1.4. Since |f (x)| ≤ |x|α for all x ∈ I, f (0) = 0. Thus this function has no limit as (x, y) → (0, 0). Since E ⊆ f (X), choose a ∈ X such that x = f (a). d) Since xn → 1/2 as n → ∞, lim supn→∞ xn = lim inf n→∞ xn = 1/2 by Theorem 2.36. Let ² > 0 and choose M xn → ∞ as n → ∞, choose N ∈ N such that n ≥ N implies xn f (xn) → L as n → ∞. 130 Copyright © 2010 Pearson Education, Inc. ∂xn ∂x1 ∂xn−1 Fx1 Fx2 Fxn 11.6.8. By Theorem C.5, · ∂f1 /∂x Df = ∂f2 /∂x ∂f1 /∂y ∂f2 /∂y ¸ implies (Df)−1 = 1 ∆f · ∂f2 /∂y −∂f2 /∂x ¸ −∂f1 /∂y . Hence by Theorem 4.17i, f increases on [4, ∞). Since f is a uniform limit of continuous functions, f is continuous on [a, b]. Since A is nonempty, it follows from the Completeness Axiom that A has a supremum. Let k ≥ N and x ∈ R. Therefore, c f (x) dx ≥ ²0 (d − c) > 0, a contradiction. 1.5.4. Suppose x belongs to the left side of (16), i.e., x ∈ X and x ∈ / ∩α∈A Eα . j≥n k≥n k≥n Taking the limit of this inequality as n → ∞ establishes a). Hence DR f (x0) = limh→0+ (f (x0 + h) − f (x0))/h ≤ 0. 9.4.10. Since ¯ ¯ µ ¶k ¯ ak+1 ¯ 2p ¯ ¯ = ≥ 2p k + 1 → ¯ ak ¯ k e P∞ p as k → ∞, by L'Hˆopital's Rule, k=1 2kp k!/k k converges absolutely when √ 2 < e, i.e., when p < log2 (e), and diverges when p > log2 (e). Finally, if x < y and 0 < b < 1, then −x > −y and by b), bx = (1/b)−x > (1/b)−y = by . 14.5.3. If SN → f and TN → f as N → ∞, then S − T is a trigonometric series which converges to zero. All rights reserved. h→0 h→0 h Thus f 0 (0) = 0 exists. Then f takes [−1, 1] onto [−1, 1] and f (0) = 1, but f −1 (f (0)) = f −1 (1) = {0, 1}. Therefore, this series converges if and only if p > 1. b) By the Squeeze Theorem, x cos(1/x2) → 0 as x → 0. 2x x 30 Copyright © 2010 Pearson Education, Inc. x+1 c) Let E = B2 (0, 0) \ B1 (0, 0). By assumption vi), 0 ≤ cos x ≤ sin x/x ≤ 1. 46 Copyright © 2010 Pearson Education, Inc. Let c := min{r0 , s0 } and d := max{2r, 2s}. If E contains two points, say a, b, then a, b ∈ Eα for every α ∈ A. 8.2.4. a) The columns of B are T (e1) = (0, 1, 1, 1), T (e2) = (0, 1, 0, 1), T (e3) = (0, 0, −1, 0), and T (e4) = (0, 0, 0, 1). 1 5.3.3. a) If x = tan θ then dx = sec2 θ dθ so Z 1 Z π/4 x3 f (x2 + 1) dx = 0 tan3 θ sec2 θf (sec2 θ) dθ. c) Since cos2 x − sin2 x = cos(2x), it follows from Example 7.44 that cos2 x − sin2 x = ∞ X (−1)k (2x)2k k=0 (2k)! = ∞ X (−4)k x2k (2k)! k=0 for x ∈ R. , 0), then by definition, x·y (b − a)2 cos θ = =√ . By looking at the graph, we see that f (E) = [19, ∞). Let P = {t0 , . To prove the left-most inequality, repeat the steps above, using part a) in place of Remark 6.22i, but with infimum in place of supremum and r1 < r in place of r0 > r, proves part c). Therefore, the solution is √ (1/3, (3 − 1)/2) ∪ (1, ∞). If x, a ∈ (0, 1) and |x − a| < δ, then |f (x) − f (a)| ≤ |x(sin 2x − sin 2a)| + |(x − a) sin 2a| ≤ 2| sin(x − a)| + |x − a| ≤ 3|x − a| < 3ε = ε. b) The ratio of consecutive terms of this series is (2k + 1)/(2k + 4) which converges to 1 as k → ∞. A similar argument works for the case t < t0 . Since X is complete and satisfies the Bolzano–Weierstrass Property 10.4.7. a) Since both sets are nonempty and ρ(x, y) is bounded below by 0, the dist (A, B) exists and is finite. j=1 Since each Braj (aj) contains xk for only finitely many k's and xk ∈ H for all k ∈ N, it follows that N is finite, a contradiction. Pk 14.2.1. Let ² > 0 and set Sk (x) := j=0 fj (x) for k ≥ 0 and x ∈ E. 10.1.9. a) Repeat the argument of Remark 10.9 with ε/2 in place of ε. Since F = ∇f , f must be C 2 on E, hence Qx = fyx = fxy = Py on E. Thus 1/yn → 0 as n → ∞ and xn ≥ 1 for all n ∈ N. Thus 5 < 2x + 3 < 7 and 0 < x − 1 < δ. Choose N ∈ N such that |ak | ≤ 1/2 for k ≥ N . If λ√= 0 then x = 2y and the constraint implies y = ±1/ 5. Thus T (1, 0, 0) = T (0, −1, 1) + T (1, 1, −1) = (1, 0) + (1, 2) = (2, 2). ¯ ¯ 0 ¯ 4 2a 4ae 2a 2a e 2 a It remains to prove the claim. 1 b) By the Chain Rule d F (x) = dx ÃZ 0 Z x3 f (t) dt − 0 ! x2 f (t) dt = 3x2 f (x3) − 2xf (x2). PN0 7.1.10. , BN , where u(x, t) ≥ −² on U := ∪N j=1 Bj . Since −1/x → −∞ as x → 0+ implies that e−1/x → e−∞ = 0 =: f (0), it follows that f is continuous on [0, 1]. ¯1 R1 b) If p 6= 1 then 0 dx/xp = x1−p /(1 − p) ¯0 . a) Suppose that H is compact and xk ∈ H. 82 Copyright © 2010 Pearson Education, Inc. b) f −1 (−1, 1) = [0, 1) is relatively open in [0, ∞), the domain of f as Theorem 9.26 says it should; f −1 [−1, 1] = [0, 1] is relatively closed in [0, ∞) as Exercise 9.4.5a says it should. 11.3.3. By Theorem 11.22, the normal direction is given by (−2x, −2y, 1). We will choose two planes with normal (0, 0, 1), e.g., z = 0 and z = 1. , fm (x)) − (b1 , . If θ is the angle between this longest side and the "first" side of Q, y := (a − b, 0, . On the other hand, if y ∈ / E, then fx0 j (y) = 0 for all j. By Remark 1.41, the unit interval (0, 1) is uncountable, hence {x : x ∈ (0, 1)} is an uncountable collection of pairwise disjoint nonempty sets which covers the unit interval (0, 1). , fxxxx = y 4 exy , fxxxy = (2+4xy +x2 y 2)exy , fxxyy = (3x2 +x3 y)exy , and fyyyy = x4 exy . If f (x0) 6= 0 for some x0 ∈ [a, b], then by part a), |f (x)| dx > 0, a contradiction. Thus |f (x) − f (y)| ≤ |f (x) − f (N)| + |f (N) − L| + |f (y) − L| 0 choose δ > 0 such that x, y ∈ E and |x − y| < δ imply |f (x)−f (y)| and |g(x)−g(y)| < ²/2. u d) Using the substitution u = t/x, we obtain Z x Z xy Z y dt dt du L(xy) = + = L(x) + L(x) + L(y). 9.5.3. Since E is compact, it is bounded by the Heine-Borel Theorem. Let y ∈ E. Let Qj be cubes such that x ∈ Qoj and Vol (Qj) → 0 as j → ∞. c) By the Archimedean Principle, given ε > 0 there is an N ∈ N such that N > 3/ε. In particular, the smooth parametrizations (φxj , Exj) satisfy S = ∪N j=1 φxj (Exj). c) The graph of f is a parabola whose absolute maximum is 1 at x = 1. If we set H := {(0, y) : −1 ≤ y ≤ 1} then E ∪ H is closed and bounded, hence compact. k=1 (kq) Pn 6.2.9. If sn := k=1 ak converges then so does s2n+1 . p (2k)! (2k)! (2k)! k=0 k=0 k=p+1 Then (*) (−1)p+1 (2p)!q/p − (−1)p+1 (2p)! p X (−1)k /(2k)! = k=0 ∞ X (−1)k+p+1 (2p)!/(2k)!. Conversely, if E ∩ Bs (a) \ {a} is always nonempty for all s > 0 and r > 0 is given, choose x1 ∈ E ∩ Br (a). Thus f is strictly increasing, hence 1–1, and ⌈ ⌈ { (x − 2)/3 f −1 (x) = x−2 | ⌊ (x + 2)/3 x≤2 2 4, 6 Copyright © 2010 Pearson Education, Inc. 1.6.1. The function 2x − 1 is 1–1 and takes N onto {1, 3, 5, . Thus P2n (x) = n X (−1)k x2k k=0 (2k)! . Conversely, if |xn | < C for all n ∈ N, then xn is bounded above by C and below by −C. d) Set F (x, y, z) = x + y + z + g(x, y). 0 13.3 Surfaces. We have shown that [α, β] ⊆ J. P∞akj < ∞. Since | sin θ| ≤ 1 no matter what θ is, we have 2 |f (x) − L| = |x3 sin(ex)| < δ 3 · 1 = ε for every x which satisfies 0 < |x| < δ. 8.3.9. Suppose E is closed and a ∈ / E, but inf x∈E kx − ak = 0. Hence by Stokes's Theorem and Theorem 7.10, ZZ Z lim curl Fk · n dσ = lim Fk · T ds k→∞ k→∞ ∂S S Z ZZ = F · T ds = curl F · n dσ. ∈ R such that x > M implies |f (x) − L| < ². On the other hand k=0 2k = 2n − 1 by induction. dx d) Let u = x − t so du = dx. Hence E = A ∩ Y, where A = Y \ U is closed in Y. We may suppose that B = +∞. But s < r, so Bs (a) ⊂ Br (a). xn = 1/2 − 1/n is strictly increasing and |xn | ≤ 1/2 < 1 + 1/n, but xn → 1/2 as n → ∞. x ∈ Ac . Thus |ak | ≤ 1/k for large k. 1.3.9. By induction, 2n > n. Thus by the Ratio Test, this series converges when 1 (i.e., when −3 < x < −1) and diverges when P∞ |x + 2| < √ |x + 2| > 1. y does not depend on the sequence xn . Hence by the Heine Borel Theorem, it is not compact. b) By a), lim inf (xn yn) = − lim sup(−xn yn) ≥ − lim sup(−xn) lim sup yn = lim inf xn lim sup yn . Hence |αxn − αxm | ≤ |α| |xn − xm | < ε for n, m ≥ N . b) Choose a < b0 < b such that f (x)/g(x) > M := min{L/2, 1} for all b0 < x < b. 10.6.6. a) Since f is continuous, so is kf k (modify the proof of Exercise 3.1.6). a) Since F (a, b) = n X ∞ X |xj | < ∞. d) Let x, y ∈ R. 3 13.2.4. a) Since τ 0 (u) = δ > 0, (ψ, J) and (φ, I) are orientation equivalent by Definition 13.18. Hence x0 ∈ (c, d). Thus ZZ Z 1Z 1 14 ω= (x, y, x4 + y 2) · (4x3 , −2y, 1) dx dy = − . Then 0 = f 0 (x) = xα−1 (α − x)/ex implies x = α. Moreover, since B = ∞, we know that f 0 (x) cannot be zero for large x. 6.5.1. a) Let f (x) = π/2 − arctan x. c) Let x−1 k = (2k+1)π/2 for k ∈ N and notice that {xn , xn−1 , . Since f is increasing, L − ² < f (x0) ≤ f (x) ≤ L for all x0 < x < b. If x > 0 and y > 0, then by the Additive Property, x + y > 0 and by the First Multiplicative Property, xy > 0. Since f is 1–1 from A onto B0 , it follows from Theorem 1.30 that x = f (f −1 (x)) = f (f −1 (y)) = y. If n = 1 then f 0 (x0) = 1 for all x0 ∈ R. When x = −1 it diverges by the Limit Comparison Test (compare it with 1/k). Since g is differentiable at a, I1 /khk → 0 as h → 0. Since ² > 0 was arbitrary, it follows that lim supn→∞ |Bn − b| = 0. Chapter 9 9.1 Limits of Sequences. Hence f is integrable on [0, 1] and 38 Copyright © 2010 Pearson Education, Inc. k=0 b) Let ² > 0 and choose N ∈ N such that k ≥ N implies |σk − L| < ². Moreover, if g(t) = t/(t2 + x), t ≥ 1, then g 0 (t) = (x − t2)/(t2 + x)2 < 0 for all t > x. This formula holds for ` = 1. R1 c) No. If f (x) = x, then −1 f (x) dx = 0 but f (x) 6= 0 for all x 6= 0. By hypothesis, h(a) > 0 and h(b) < 0. Set f (t) = (log t + x)/(t + x) and g(t) = 1 + x/t − log t − x, for t > 1. d) 0 = fx = 2ax + by and 0 = fy = bx + 2cy imply (b2 − 4ac)y = 0, i.e., x = y = 0. Let φk be defined to be zero off the interval Ik := [2−k , 2−k+1) and be defined on Ik so its graph forms a triangle R1 with base Ik and height 2k+1 . We conclude that f is differentiable on R2 . 2.3.0. a) False. 95 Copyright © 2010 Pearson Education, Inc. Since f is convex, f (x) ≤ y ∗ . 2 b) By Definition 1.1, if a ≥ 0 then a+ = (a + a)/2 = a and if a < 0 then a+ = (−a + a)/2 = 0. Pn i.e., λ = (2/p) j=1 x2j . Hence given n ≥ N , ∞ X 0 ≤ s − sn = |ak | ≤ |aN | k=n+1 ∞ X xk−N = |aN | k=n+1 xn−N +1 . Then by Exercise 1.2.5c, 0 < xn+1 < xn . Thus we need only check whether f[0,∞) (0) = 0. 3.3.5. Since M − f (x) > 0, it follows from the Sign Preserving Property that there is an interval I centered at x0 such that M − f (x) > 0, i.e., f (x) < M for all x ∈ I. Then (1, 0) is a boundary point of E which does not belong to E. ∂gm /∂x1 (a) . 0 b) If (φ, E) is the parameterization given in Example 13.31, then kφu × φv k = k(a2 cos u cos2 v, a2 sin u cos2 v, a2 sin v cos v)k = a2 | cos v|. Since sin(1/tk) = (−1)k , it is clear that kφ(tk) − φ(tk+1)k ≥ 2 for each k ∈ N. Then √ √ √ √ F · ψ 0 = (sin t/ 3, − sin t/ 3, 0) · (− sin t, sin t, cos t/ 3) = −2 sin2 t/ 3. If k > N , then k(1/k, 1 − 1/k 2) − (0, 1)k2 = 1/k 2 + 1/k 4 ≤ 1/k 2 + 1/k 2 < ε2 . If x, y ∈ E and |x − y| < δ, then |f (x) − f (y)| ≤ |f (x) − fN (x)| + |fN (x) − fN (y)| + |fN (y) − f (y)| < ². Thus the integral converges if and only if p > 1. Thus nq is algebraic. b) By part a) and the Dot Product Rule, 0 = (ν 0 (s) · ν 0 (s))0 = 2ν 0 (s) · ν 00 (s). 0 ≤ (cb − ad)2 = c2 b2 − 2abcd + a2 d2 implies 2abcd ≤ c2 b2 + a2 d2 . Therefore, 8.2.11. 2 0 5.2.2. a) By the First Mean Value Theorem for Integrals, there is a c ∈ [a, b] such that Z 0= b Z b f (x)xn dx = f (c) a xn dx =: f (c) · I. 14.1.2. By definition and a sum angle formula, 1 2π (SN f)(x) = Z π f (t) dt −π Z N X cos kx + π sin kx f (t) cos kt dt + π −π Z π f (t) sin kt dt π −π Ã ! Z N 1 π 1 X = f (t) + (cos kx cos kt + sin kx sin kt) dt π −π 2 k=0 Z 1 π = f (t)DN (x − t) dt. 9.2 Heine-Borel Theorem. The actual value is ∆w = f (1.01, 1.98, 1.03) − f (1, 2, 1) ≈ 3.049798 − 3 = 0.049798. ¯ k k ¯ ¯ bn bm ¯ bm bm bm k≥m k≥m k≥N k=m We conclude that |bm P∞ ak | < ² for m ≥ N , i.e., bm P∞ ak → 0 as m → ∞. r2 ∂θ2 r ∂r ∂r 11.4.7. a) By the Product Rule, uxx (x, t) = ∂ ∂x µ ¶ µ 2 ¶ −2x x 1 u(x, t) = − u(x, t) = ut (x, t). Therefore, x ∈ Eαc for some α ∈ A, i.e., x belongs to the right side of (16). b) Using the substitution u = x3 , dx = 3x2 dx, we have Z 0 2 x3 x e −∞ 1 dx/(1 + x) = 3 2 Z 0 eu du = −∞ 1 . c) Set F (x, y, z) = xyz(2 cos y − cos z) + z cos x − x cos y. 1.5.0. a) False. 1.6.4. By definition, there is an n ∈ N and a 1–1 function φ which takes Z := {1, 2, . Hence by Dirichlet's Test, a (b − b) converges, say to s. 3.3 Continuity. Conversely, if f g is continuous at a and f is continuous and nonzero at a, then g = f g/f is continuous at a by Theorem 3.22. But the angle between ν 0 and ν 00 is π/2 and sin(π/2) = 1. , xN . On the other hand, by homogeneity, k(αf)(a + h) − (αf)(a) − (αT)(h)k kf (a + h) − f (a) − T (h)k = |α| khk khk Since this last term converge to α · 0 = 0 as khk → 0, it follows that αf is differentiable, and D(αf)(a) = αT . If y = e1/x then log y = 1/x, i.e., x = 1/ log y. The estimates in part b) can be made uniform if σk converges uniformly. Consequently, f −1 (f ({a})) ⊇ {a, b} ⊃ {a}, which contradicts c). If k=1 |Ik | < 1 then some point of [0, 1] is PN uncovered. Therefore, f (E) = [1, 5]. Suppose f is integrable on [a, b]. 7.4 Analytic Functions. Now yn < xn implies 2yn < xn + yn . , a) = (b − a, . Hence the integral represents the area of that semicircle, i.e., Z p 1 a2 − x2 dx = πa2 . By the Approximation Property for Infima, choose xk ∈ A and yk ∈ B such that kxk − yk k → dist (A, B). Thus by part a), (f −1)0 (e) = 1/f 0 (1) = 1/(4e). 113 Copyright © 2010 Pearson Education, Inc. Therefore, ¯Z ¯ ¯ 1 2 n−1 X Z 1 x2k ¯¯ 3 ¯ x e − ¯ ¯≤ . k∈Z 1.5.3. a) The minimum of x−2 on [0, 1] is −2 and the maximum of x+1 on [0, 1] is 2. We conclude that f (1, −2, 0, 1) = 2 is the minimum and f (1, 2, −1, −2) = 3 is the maximum. Suppose n ≥ N and x ∈ [0, 1]. By elementary integration, it is easy to see that if fn (x) = xn , then kfn k1 = 1/(n+1) → 0 as n → ∞. Then 0 < k(x, y) − (a, b)k < δ implies |x − a| < δ, so g(x, y) − f (a)| = |f (x) − L| < ε. Vol (Br (x0)) Br (x0) 12.2.4. a) Since U (f, G) − L(f, G) ≤ U (f − fN , G) − L(f − fN , G) + U (fN , G) − L(fN , G) =: I1 + I2 and fN is integrable, we can show that f is integrable if we show I1 is small. Then both terms on the right side of part a) which end in −γ converge to zero as n → ∞. 2 x(x2 + z 2) dz dx = 0 1 3 Z 1 (x − x7) dx = 0 122 Copyright © 2010 Pearson Education, Inc. b) Let E = [0, 1]. n xn ¯ |xn | 2.1.3. a) If nk = 2k, then 3 − (−1)nk ≡ 2 converges to 2; if nk = 2k + 1, then 3 − (−1)nk ≡ 4 converges to 4. Since a ∈ K, we conclude that f (x) = f (a) for all x ∈ K. By part a), f (a+) and f (b−) exist. , 1). b) Define f on R by f (x) = x for x 6= 0 and f (0) = 1. , yn + xn) = y + x and x · y = x1 · y1 + · · · + xn · yn = y1 · x1 + · · · + yn · xn = y · x. C2 E b) Since Qx = (y 2 − x2)/(x2 + y 2)2 = Py , we can replace ∂E with any simple closed curve which surrounds (0, 0) and is disjoint from ∂E. 13.1.10. If it's true for some n ≥ 1, then by the Inductive Hypothesis, definition, and the Sum Rule, (f + g)(n+1) = (f (n) + g (n))0 = f (n+1) + g (n+1) . Then f (x) → 0 as x → 0, but g(x, y) = f (x)(y + 1) does not have a limit as (x, y) → (0, 0). Thus F = (P, Q) = (fx , fy) is conservative by definition. 0 ¯e By hypothesis, either f (0) = 0 or g(0) = 0, and either f (e) = 0 or g(e) = 0. b) By the Quotient Rule, µ ¶0 f f 0 (3)g(3) − f (3)g 0 (3) 2b − d (3) = = . Then by Exercise 1.2.5a, 2 < xn+1 < xn . If x < M , then x + 1 < −1/ε < 0 so |x + 1| = −(x + 1) > 1/ε. Choose a partition {x0 , x1 , . If p ≤ 0, this series diverges by the Divergence Test. Then xn , yn → ∞ so choose N so large that Q(xN , yN) > 2/δ. 3.2.2. a) x3 −x2 −4 = (x−2)(x2 +x+2) and x2 −4 = (x+2)(x−2) so (x3 −x2 −4)/(x2 −4) = (x2 +x+2)/(x+2) → 8/4 = 2 as x → 2−. R∞ ∞ c) 0 eat e−st dt = 1/(s − a) for s > a. In particular, |x0 − t0 | ≤ c − x0 < δ. Since [Bα = E ∩ α∈A [Vα , α∈A and the union of Vα 's is open by Theorem 8.24, it is clear that the union of the Bα 's is relatively open in E. But by Lindel¨of (or using rational centers and rational radii as in the proof of the Borel Covering Lemma), we can find open balls Bj := B²j (xj) ∞ ∞ such that V ⊆ ∪∞ j=1 Bj . Hence it surely converges. Moreover, by L'Hˆopital's Rule, log(1/f (k)) f 0 (k)/f (k) = − lim ≡ −α. b) Since x2 − 2x + √ 3 > x2 2 2 implies x < 3/2, inf E = 0, sup E = 3/2. By Theorem 10.16, E must be closed. Either there is an a ∈ H such that for each r > 0, Br (a) contains xk for infinitely many k's, or for each a ∈ H there exists an ra > 0 such that Bra (a) contains xk for only finitely many k's. If f is continuous at x = 1 then |f (x0) − f (yn)| = |f (x0 /yn)| → |f (1)| = 0 as n → ∞, i.e., f is continuous at x0 . x0 < 1. n→∞ n→∞ b) It suffices to prove the first identity. 3 3 2 3 3 2 1 + t (1 + t) 1 + t (1 + t) (1 + t3)2 148 Copyright © 2010 Pearson Education, Inc. 10.3.5. Suppose E is closed and a ∈ / E. Since DE > 0 and c 6= 0, it follows from Theorem 11.61 that the point (x, y, z) identified above is a maximum when Fxx /2 = cD < 0 and a minimum when cD > 0. It follows that G(f) is of volume zero. Let ψ(x) := f (φ(x)) for x ∈ Z. Similarly, y ∈ Bs (b). d) Let s = E(x), t = E(y), and w = E(x + y). This last series converges since α > 1/2. 2a √ We claim that e ≤ 2/(ae). a Thus 7.50 (with x0 = 0), Rnf,0 (a) → 0 as n → ∞, for all a ∈ R. 2 c) −1/x2 → 0 as x → −∞ so e−1/x → e0 = 1. It can be parameterized by φ(t) = (cos t/ 2, cos t/ 2, sin t), t ∈ [0, 2π]. By part a), V = ka × bk · h, where h = kck · | cos θ|. By the first derivative test, φ(1) = 1/e is a local maximum. 9.5.4. Suppose A is uncountable. Since {xn } is increasing, n ≥ N implies xn ≥ xN > M . 2.5.1. a) Since 3 − (−1)n = 2 when n is even and 4 when n is odd, lim supn→∞ xn = 4 and lim inf n→∞ xn = 2. Since n/π → ∞ as n → ∞, we conclude that φ is not of bounded variation on [0, 1]. b) If xn → a in the discrete space, then for n large, σ(xn , a) < 1. 18 12.3.5. a) If f is continuous on R then f is integrable on R by Theorem 12.21, f (·, y) is integrable on [a, b] by Theorem 5.10, and f (x, ·) is integrable on [c, d] (also by Theorem 5.10). Taking the limit of (*), as kj → ∞, we see that kb − ak ≤ M , i.e., b ∈ B. Since ¯ ¯ ¯ f (0 + h) − f (0) − ∇f (0) · h ¯ |f (h)| ¯ ¯ = ≤ khkα−1 → 0 ¯ ¯ khk khk 103 Copyright © 2010 Pearson Education, Inc. Thus |f (x) − L| = |x − 1| < δ = ε for every x which satisfies 0 < |x − 1| < δ. 7.2.3. a) Since |xk+1 /(k + 1)!|/|xk /k!| = |x|/(k + 1) → 0 as k → ∞, this series converges pointwise on R by the Ratio Test. √ 2.2.5. Case 1. Thus by the calculations p in part b), choose n = 14. 2.3.5. The result is obvious when x = 0. Hence integrating term by term, we obtain µ ¶ Z π/2 Z π/2 ∞ X X 1 1 kπ f (x) dx = cos(kx) dx = sin . Since g is nonnegative, we have f (x) < (2L + 1)g(x) for x ∈ (b0 , b). c) The columns of B are T (e1) = (1, −1), T (e2) = · · · = T (en−1) = (0, 0), and T (en) = (−1, 1). 94 Copyright © 2010 Pearson Education, Inc. Copyright ©2010 Pearson Education, Inc. By definition, ZZ Z 1 Z 1−y ω=− S1 ZZ Z 1 Z ZZ P (0, y, z) dy dz, 0 0 1−z ω=− S2 R(x, y, 0) dx dy, 0 Z and 0 1 Z ω=− S3 1−z Q(x, 0, z) dx dz. For example, Z 1 π (f (t) + g(t)) cos kt dt π −π Z π Z 1 1 π = f (t) cos kt dt + g(t) cos kt dt = ak (f) + ak (g). 73 Copyright © 2010 Pearson Education, Inc. If x ∈ [−M, M] then |fk (x)−f (x)| < ². c) True. To show E ⊆ f −1 (f (E)), let x ∈ E. Since f 0 = −g 0 , it follows that f 0 (x1) < 0 and f 0 (x2) > 0. 8.2.1. a) By definition, a − b and a − c lie in the plane. 11.3.1. Since they are all C 1 on their domains, they are all differentiable on their domains. By the choice of δ, it follows that |²h (x)| ≤ ²khk. Since the x axis lies to the left of the yz plane, we can parameterize this curve by φ(t) = (3 sin t, 0, 3 cos t), I = [0, 2π]. 4.3.6. By the Mean Value Theorem, f (c) = f (c) − f (a) = (c − a)f 0 (x1) and f (c) = f (c) − f (b) = (c − b)f 0 (x2) for some x1 , x2 ∈ (a, b). 58 Copyright © 2010 Pearson Education, Inc. 3.4.6. a) Suppose I has endpoints a, b. 13.6.1. a) The trivial parameterization of z = −x, x2 + y 2 ≤ 1, has normal (1, 0, 1), whose induced orientation on C is counterclockwise. Then |xn − yn | = 1/n → 0 as n → ∞, but neither xn nor yn converges. Since m ≤ f (0) < f (x) for all x ∈ / [−N, N], it follows that m is the absolute minimum of f on R. 11.4.1. By the Chain Rule, ∂w ∂F ∂x ∂F ∂y ∂F ∂z = + + , ∂p ∂x ∂p ∂y ∂p ∂z ∂p ∂w ∂F ∂x ∂F ∂y ∂F ∂z = + + . Since xk ∈ E too, it follows that xk ∈ U for large k. Taking the limit of √ below. Since z ≥ 0, the solution with√both minus signs is extraneous. 4.5.8. By 4.5.6a, f −1 is differentiable on [a, b]. Then both f and g are strictly decreasing, hence 1–1 on R. It follows from Lemma 1.40 that A is at most countable, a contradiction. b) By Gauss' Theorem and Exercise 13.5.8, ZZ ZZZ ZZZ u∇v · n dσ = ∇ · (u∇v) dV = (∇u · ∇v + u∆v) dV. If f (x) < 1, then f (x) < 1 ≤ 1 + f (x). Since in a bounded interval, E is bounded. c) If ∇f (a) = 0 then Du f (a) = 0 and there is nothing to prove. On the other hand, since d/dt(te−(s−a)t) = e−(s−a)t (1 − (s − a)t), integration by parts yields Z ∞ L{tf (t)}(s) = Z −t(s−a) te f (t)e −at ∞ dt = − 0 e−(s−a)t φ(t)(1 − (s − a)t) dt. Similarly, if rx ∈ Q and r 6= 0, then x ∈ Q, a contradiction. 3.4 Uniform continuity. , N }, then by repeating the proof of part b), we can choose smooth curves {(ψj , [(j − 1)/N, j/N]) : j = 1, . Pn−1 Pn−1 6.1.10. 0 3 k(1, f (t), 0)k (1 + (f 0 (t))2)3/2 13.2 Oriented Curves. Therefore, −Fxn−1 ∂g1 ∂g2 ∂gn −Fxn −Fx1 ··· = ··· = (−1)n . But k k k=1 P k=1 ak bk converges, so we can add it P∞ P∞ to both sides of s = k=1 ak (bk − b) ≡ k=1 ak bk − k=1 ak b. b) Since sin2 hx is increasing on [2n−1 , 2n], sin2 kh = sin2 kπ π 1 ≥ sin2 = 2n+1 4 2 for k ∈ [2n−1 , 2n]. Hence by the Weierstrass M-Test, the j-th term by term derivative of Sf converges uniformly on R. Then div F = 0. Hence ¯ ¯ ¯ ¯ ¯ ∂f ¯ ¯ ¯ ¯ ¯ ¯ = lim ¯ f (0, 0, . Then there is an ² > 0 such that B² (a) ∩ E = ∅. 9.3.8. a) Repeat the proof of Remark 3.4, replacing the absolute value by the norm sign. e) By assumption v), part d), and the Chain Rule, (cos x)0 = − cos(π/2 − x) = − sin x. 8.1.5. Let a, b, c denote the vertices of ∆, and C be the line segment between a and b. Since f (15) = 0.0068, n = 15 terms will estimate the value to an accuracy of 10−2 . 8.1.3. It is clear that equality holds if either x = 0 or y = 0. Since |x| is periodic and continuous on [−π, π], it follows from Theorem 14.29 that this series converges to |x| uniformly on [−π, π]. 9.5.5. By Exercise 8.3.8, there exist ² := ²x > 0 such that V = ∪x∈V B² (x). 4 13.5.2. a) By Green's Theorem, Z Z d Z b ω= C (y − 1) dx dy = (b − a)(c − d)(c + d − 2)/2. c) Using part b) and the product rule, (f /g)0 = (f · 1/g)0 = f 0 · 1 g0 gf 0 − f g 0 −f 2 = . 11.5.11. By definition, then, xn → a as n → ∞. 1.2.10. (See the argument which appears in c) above.) 10.2.2. a) If a is not a cluster point, then some Br (a) contains only finitely many points of E \ {a}, say x1 , . If k := j − 1 ≥ 0 then 0 ≤ k < j. Thus the original series diverges when 0 < q ≤ 1 and converges when q > 1. kuk2 In particular, the transitions are C p on g(U ∩ V). Since φ0 (θ) = (f 0 (θ) cos θ − f (θ) sin θ, f 0 (θ) sin θ + f (θ) cos θ) we have kφ0 (θ)k2 = |f 0 (θ)|2 + |f (θ)|2 6= 0. Since E and A are closed, it follows that C is closed. Moreover, it is obvious that they are nested. b) If f (b) < f (a) then by part a), given y0 ∈ (f (b), f (a)) there is an x0 ∈ (a, b) such that y0 = f (x0) and DR f (x0) ≤ 0. Then (x − ², x + ²) ∩ E contains infinitely many points, so x is a cluster point of E. If (xk , yk) ∈ R converges to some (x, y) then a ≤ xk ≤ b implies a ≤ x ≤ b and similarly, c ≤ y ≤ d. By a similar argument, if k2 > r1 is least such that sk2 > y, then sr1 < x ≤ y < sk2 + bk2 for all r1 ≤ ` ≤ k2 . 6.3.0. a) True. Also f + − f − = 2f /2 = f and f + f − = 2|f |/2 = |f |. Hence, if x, y ∈ R and kx − yk < δ, then (*) kT (x, y)k ≤ M ² · kx − yk. Since 2x3 − 3x + 1 = (x − 1)(2x + 2x − 1) implies that x = 1, (−1 ± 3)/2, √ √ the second inequality is equivalent to (−1 − 3)/2 < x < (−1 + 3)/2 or x > 1. a Rb Rc Rb Thus (U) a f (x) dx ≥ (U) a f (x) dx + (U) c f (x) dx. d) Integrating by parts twice, Z ∞ Z Z 1 b2 ∞ −st 1 b2 ∞ −st e−st cos bt dt = − e sin bt dt = − 2 e cos bt dt s s s 0 s s 0 0 R∞ for s > 0. n a 7.1.9. a) By the Extreme Value Theorem, f is bounded on [a, b] and there are positive numbers ²0 and M such that ²0 < |g(x)| < M for all x ∈ [a, b]. It follows from the Limit Comparison Test that we need only consider k=1 (kq k)−1 . Since | cos(n2 + n + 1)| ≤ 1, it follows that {xn } is bounded. b) By Theorem 4.33, f −1 is differentiable on (c, d) and (f −1)0 (x) = 1/f 0 (f −1 (x)). c) By Theorem 12.7 we must show that Vol (E1 ∪ E2) ≥ Vol (E1) + Vol (E2). Since the ratio of successive coefficients is (2k − 1)/(2k + 2) = 1 − (3/2)/(k + 1), it follows from Raabe's Test that the series converges absolutely at both endpoints. If k > 0 then nq is a root of the polynomial xj − nk . Then we can compute cos θ two ways: |w · (a, b, c)| h = | cos θ| = . Thus by theorem 10.52, g is uniformly continuous on E. u2 2 c) Since div (x, y, z) = 3, we have by Gauss' Theorem that Z 1 Vol (E) = x dy dz + y dz dx + z dx dy. Define g(x) := limn→∞ f (xn). Modify the proofs of Remark 2.4, Theorems 2.6, 2.8, and Remark 2.28 by replacing the absolute value signs by the metric ρ. By definition, there is a partition P² of [a, b] such that U (f, P²) − L(f, P²) < ². b) The same estimates can be obtained. By Exercise 8 in 8.3, E \ C is relatively open in E, i.e., E \ C = E ∩ V for some open V . Indeed, Z 1 Z 1 f (x, y) dx dy = 0 0 = ∞ Z X 2−k+1 −k k=1 2 Z ∞ X 2−k+1 k=1 2 Z 1 µZ 1 ¶ f (x, y) dx dy 0 µZ 1 φk (y) ¶ (φk (x) − φk+1 (x)) dx dy = 0, but Z 1 Z 1 f (x, y) dy dx = 0 ∞ Z X k=1 Z 1 2−k+1 µZ ¶ f (x, y) dy 2−k Z 1 = φ1 (y) dy dx + 1/2 dx 0 0 ∞ Z X k=2 µZ 2−k+1 2−k φk (x) ¶ 1 (φk (y) − φk+1 (y)) dy dx 0 1 = 1 + 0 = 1. It is relatively closed in B√2 (2, 0) because the limit of any convergent sequence (in the SUBSPACE sense) in the set stays in the set. Thus Rx = y + fx and we may set f = 0, i.e., R = xy. u 2 (4 + 4x − 3x2) dx = 1. Thus the integral converges if and only if p < 1. Differentiating term by term, we obtain f 0 (x) = ∞ X k=1 Rx ∞ X kxk−1 kxk−1 = =: g(x) ((−1)k + 4)k 3k k=1 P∞ for 0 ≤ x < 3. 10.4.2. Let A, B be compact sets. 10.5.5. Suppose A is not connected. Therefore, ur = gy cos θ − gx sin θ = vθ /r and vr = −fy cos θ + fx sin θ = −uθ /r. 2 d) The boundary of S is given by 2x2 + z 2 = 1, y = x, oriented√in the counterclockwise direction when viewed √ from far out the x axis. 3 137 Copyright © 2010 Pearson Education, Inc. 0 c) The boundary√of S is given by x2 + y 2 = 10, z = 0, oriented in the clockwise direction. But Eα is an interval, hence (a, b) ⊂ Eα for all α ∈ A, i.e., (a, b) ⊂ E. Hence by the Second Multiplicative Property and Theorem 1.20, inf E + ² = −(sup(−E) − ²) > a > − sup(−E) = inf E. Since f (t) ≤ F (x0) for all t ∈ [a, x0], it follows that t0 ∈ (x0 , c]. From part a), we see that Pn this function has only one critical point: (a0 , b0). By the Mean Value Theorem, f (n + 1) − f (n) = f 0 (cn) for some cn ∈ (n, n + 1), n ∈ N. We conclude that f (−2, 0) = −2 is the minimum, f (1/2, ± 15/2) = 17/4 is the maximum (and f (2, 0) = 2 is a saddle point). b) Let ² > 0 and x ∈ [a, b]. If a = b then a + c = b + c since + is a function. Therefore, it suffices to prove that given any continuous f on [a, b], there is a polynomial Q, with rational coefficients, such that |f (x)−Q(x)| < ε for all x ∈ [a, b]. x √ √ This last quotient converges to 0 by Theorem 2.12. Thus V (E; Gm) = 2m · 22m = 2−m < ε. Since c is a point of discontinuity, it follows that f 0 (c−) < f 0 (c+). 10.4.6. Suppose that f is uniformly continuous on E. , xN such that N [X= Bxj . To prove this, combine the inequality in part b) above with the Comparison Theorem for improper integrals. It is clear by construction that f is 1–1. Since the geometric series k=0 (r0 /r) < ∞, it follows from the Comparison Test that k=0 ak r0 converges, which contradicts the fact that r0 > R and R is the radius of convergence. If y = 0√then the constraint implies x = ±2. Thus the inequality holds for all k ∈ N. 9.3.1. a) The domain of f is all (x, y) ∈ R2 such that x 6= 1 and y 6= 1. d) The set is closed but not bounded (since (n, 1/n) ∈ E for all n ∈ N). Hence by Theorem 10.56, f (E) is an interval. 11.5.2. We must show D(`) f ((a, b); (h, k)) = ¯ µ ¶ X ¯ j=0 ∂`f ∂xj ∂y `−j j (a, b)hj k `−j for ` ∈ N. c) Let a = 0 and b = 1. In particular, the chord from (x1 , f (x1)) to (x2 , f (x2)) lies on or above the graph of y = f (x), i.e., f is convex. b) Let xn = n and yn = n + 1/n. Rx 14.4.3. By the proof of Corollary 14.27, if a0 (f) = 0 then F (x) := 0 f (t) dt is continuous, periodic, of bounded variation, ak (F) = bk (f)/k, and bk (F) = −ak (f)/k = 0 for k ∈ N. Thus ∩x∈(0,1] [x − 1, x + 1] = (0, 1]. 11.7.5. If fxy (a, b) 6= 0 then fxx (a, b) = fyy (a, b) = 0 and it follows that D(2) f (a, b) = fxy (a, b)hk takes both positive and negative values as h, k range over R. √ √ b) Clearly, fx = 1/(2 x), fy = 1/(2 y), fxx = −1/(4x3/2), fxy = 0, fyy = −1/(4y 3/2), fxxx = 3/(8x5/2), fyyy = 3/(8y 5/2), and all mixed third order partials are zero. When p < 0, the result is false, since ak = 1/k 1−p generates a convergent series by the p-Series Test (1 − p is GREATER than 1 in this case), but |ak |/k p = 1/k which generates the harmonic series, which diverges. In particular, it follows from assumption ii) that (1 − cos x)/x → 0 as x → 0. 5.1.4. a) If f (x0) 6= 0 then given ² > 0 choose by the Sign Preserving Property a δ > 0 such that |f (x)| ≥ ² for |x − x0 | ≤ δ. Thus f is continuous on R2 . 8.2.5. a) T (1, 0) = T (1, 1) − T (0, 1) = (−1, π, −1), and T (0, 1) = (4, 0, 1). Let ² > 0 and choose N so large that k=N +1 |ak |/k < ²/2. If N > N2 then |(S0 (x) − f (x)) + · · · + (SN (x) − f (x))| N +1 µ ¶ N1 X 1 ² N − N1 ² ² ≤ |Sk (x) − f (x)| + < + = ². , jn . Since wxx (c, t2) − wxx (x2 , t2) → 0 as h → 0, and w(x2 , t2) ≤ w(x2 + h, t2) (this point is a local minimum), it follows that wxx (x2 , t2) ≥ 0. √ Note: If we replace φ by g(x) = sin x − 2 2x/π, then the same argument shows g(x) ≥ 0 for x ∈ [0, π/4], and we obtain ¯Z ¯ π ¶ ¯ π/2 ¯ π √ π π 1 ¯ ¯ −a sin x e dx¯ ≤ e−a 2/2 + √ ≤ √ + 1, ¯ ¯ 0 ¯ 4 2a 2 2a 2 e an improvement over the estimate we already obtained. If n > 9 then (n + 3)2 = n2 + 6n + 9 < n22 + 7n < n2 + 8n + 16 = (n + 4)2 . If x < y belong to [a, b], then f (x) ≤ f (y) and g(x) ≤ g(y). For p = −1/e, the series converges conditionally by the Alternating Series Test and what we just proved. u→∞ 1 u Hence the series diverges by the Logarithmic Test. ∂x∂y ∂p ∂p ∂x∂z ∂p ∂p ∂y∂z ∂p ∂p 11.4.2. a) By the Chain Rule ⌈ ∂h/∂x1 (a) . 10k 10k 10 sn := k=1 k=1 Then 0 < x − sn < 1/10n , so by a) choose xn+1 such that xn+1 /10n+1 ≤ x − sn < xn+1 /10n+1 + 1/10n+1 , i.e., n+1 X k=1 n+1 X xk xk 1 ≤ x < + n+1 . 10.3.11. E 13.4.5. By definition, ZZZ Z 1 Z 1−z 1−y−z Px dV = Px dx dy dz E Z 0 0 1Z 0 1−z = (P (1 − y − z, y, z) − P (0, y, z)) dy dz. As x ranges over E, the bk 's exhibit all possible combinations of 0's and 2's, hence the binary coefficients of f (x) exhibit all possible combinations of 0's and 1's. j=1 Let x ∈ K. If not, then some xn satisfies |f (xnk)| > ε0 for k ∈ N, so |f (xnk)|/(xnk − a) > ε0 /(xnk − a). 12.5.2. Products of C ∞ functions are C ∞ functions. 8.3.6. a) If C is relatively closed in E, then there is a closed set A such that C = E ∩ A. If it holds for some n ≥ 3 then by a), (n + 1)2 = n2 + 2n + 1 < 2n + 2n = 2n+1 < 2n+1 + 1. e) Let f (x) = π/2 − arctan x. If x0 = 0, then xn = 0 for all n. 2 2 b) If x = sin θ then dx = cos θ dθ so Z 3/2 0 √ Z π/3 x3 f (1 − x2) dx = 0 sin3 θ cos θf (cos θ) √ dx = dθ. Similarly, since k1 = a− k = ∞ we can choose an integer r1 > k1 least such that sr1 := b1 + b2 + · · · + br1 ≥ sk1 − a− 1 − · · · − ar1 −k1 < x, and sr1 ≥ x + br1 . Printed in the United States of America. x∈V 92 Copyright © 2010 Pearson Education, Inc. c) Let a = −1, b = 1, and f (x) = x3 . Since 1/n → 0 as n → ∞, it follows that xn satisfies the hypotheses of Exercise 2.4.4. Hence xn must converge to a finite real number. Since (|t|p)0 = p|t|p−1 exists forP all t ∈ R and n p > 1, it follows from Lagrange's Theorem that if f (x) is an extremum subject to the constraint k=1 |xk |p = 1 p−1 2 p then 2xj = p|xj | λ. Thus E is sequentially compact. c) This set is bounded but not closed. 13.1.1. (ψ, I) runs clockwise. Then f (x, y) = x2 + 2x − y 2 = 5 cos2 θ + 4 cos θ − 1 =: h(θ). k=n+2 (2ak /x Ã N X Moreover, by the claim and (*), ! k −1/x2 (ak /x)e −0 = 0. β) Since f (x) = x2 is increasing on [0, ∞), Mj = x2j := ((j/n)2 and mj = x2j−1 := ((j − 1)/n)2 . 5.1.0. a) False. Let y ∈ Br (x) and let P be a polygonal path from x0 to x which lies in E. Since f (x) ≥ c > 0 for all x ∈ E and x ∈ [xj−1 , xj],then f 1/m (x) − f 1/m (y) ≤ Mj (f) − mj (f) Mj (f) − mj (f) = Cm (m + 1)c(m−1)/m 45 Copyright © 2010 Pearson Education, Inc. It follows that 1/f is differentiable at a and its derivative is T . Choose δ > 0 such that |φ(t)| < ² for 0 ≤ t < δ. Hence, √ √ (P, Q, R) · φ0 (t) = (cos t sin2 t/4, − cos t sin2 t/2, 0) · (− sin t, cos t/ 2, cos t/ 2) = − sin3 t cos t/4 − sin2 t cos2 t/(2 2). Then f 0 (x) = 3x2 + 1 > 1 for all x ∈ (0, ∞), but x2 /f (x) < 1/x → 0 as x → ∞. Finally, √ xn+1 1 − 1 − xn 1 − (1 − xn) 1 1 √ = = = . 11 Copyright © 2010 Pearson Education, Inc. R1 as n → ∞, 0 x2 dx = 1/3. Thus the Taylor polynomials fn of f converge to f uniformly on [a, b] ⊂ I. xn−1 + yn−1 = 1 > 0, so xn > xn yn−1 = yn > 0 1 · 1 = 1. Substituting 2x for x, we have cos(2x) = x2 + cos(2x) = 1 − x2 + ∞ X (−4)k x2k (2k)! k=2 for x ∈ R. By definition, h/(1 + e1/h) − 0 1 = lim = 0. V is open since it is a union of open intervals. Nevertheless, it is clear that |x4 − π| < 0.0000000005 which is much smaller than 0.000136465. b) dz = y cos(xy) dx + x cos(xy) dy. 13.5.3. a) By Gauss' Theorem, ZZ Z 3 Z 2 Z 1 F · n dσ = S 0 0 (2 + ez) dx dy dz = 2(5 + e3). c) Rationalizing the expression, we obtain √ 3n + 2 − √ n= √ √ √ √ (3n + 2 − n)(3n + 2 + n) 2n + 2 √ =√ √ √ → ∞ 3n + 2 + n 3n + 2 + n √ as n → ∞ by the method of Example√2.13. b) The converse is false. 1−r Since rn → 0 as n → ∞, we see that xn is Cauchy, hence converges to some b ∈ I, since I is closed. Thus the formula is correct for this case. b) Repeat the proofs of Theorem 2.8 and Remark 2.28, replacing the absolute value by the norm sign. S E b) The boundary of S is given √ by x2 + y 2 = 3, z = 0, oriented in the counterclockwise direction. Let φ(t) = t log(1 − x/t). 10.6 Continuous Functions. A similar argument shows that {x ∈ Rn : kx − ak > r} and {x ∈ Rn : kx − ak < s} are both open, hence E := {x ∈ Rn : s ≤ ρ(x, a) ≤ r} = {x ∈ Rn : kx − ak > r}c ∩ {x ∈ Rn : kx − ak < s}c is closed. By the choice of δ, we conclude that F (x0) ≥ f (x0) ≥ f (t0) − ε0 /2 = F (c) − ε0 /2, i.e., F (c) − F (x0) ≤ ε0 /2 contrary to the choice of x0 . If m and m e are both infima of E then m ≤ m e and m e ≤ m, i.e., m = m. Thus f 0 (x) exists. |t1 | |t2 | kak2 |t1 | |t2 | Hence θ = 0 or π. Let ² > 0 and choose N so large that n ≥ N implies |xn | < ²2 . 14.4.5. a) Fix h ∈ R and k ∈ N. (It holds for n = 1, and if it holds for n then 6(n + 2) = 6(n + 1) + 6 ≤ 4 · 3n + 6 < 4 · 3n + 8 · 3n = 4 · 3n+1 .) Therefore, 3(n + 1)2 + 3(n + 1) + 1 ≤ 2 · 3n + 6(n + 1) ≤ 2 · 3n + 4 · 3n = 2 · 3n+1 . On the other hand, the right side lies between 1/((2p + 1)(2p + 2)) and 1/((2p + 1)(2p + 2)) − 1/((2p + 1)(2p + 2)(2p + 3)(2p + 4)), i.e., is a number between 0 and 1. 2 √ c) E = {(x, y, z) : 0 ≤ y ≤ 1, y ≤ x ≤ 1, 0 ≤ z ≤ x2 + y 2 } and Z 1 Z 1 √ (x2 + y 2) dx dy = 0 1 3 Z 1 (1 + 3y 2 − y 3/2 − 3y 5/2) dy = 0 26 . 15.3 Stokes's Theorem on Manifolds. 3 = 1.4.5. 0 ≤ an = . A similar argument works for f ∧ g. It follows from the definition of the operator norm that |f (g(x)) − f (g(a))| ≤ kDf (g(c))k kx − ak. P∞ 6.2.6. a) If an /bn → 0 then an ≤ bn for n large. 3 Vol (E) = 13.5.6. a) Parameterize ∂E by φ(t) = (cos t, sin t), I = [0, 2π]. k b) By L'H´ e−1 . b) f (x) = x is uniformly continuous on [0, ∞) but not bounded there. CHAPTER 2 2.1 Limits of Sequences. By Exercise 1.2.9c, r + 2 is irrational. Then A − B = [−1, 1] so sup(A − B) = 1 6= 0 = sup A − sup B. =° T ° kxk kxk ° 2) Taking the supremum of this last inequality over all x 6= 0, we obtain kT k ≤ M1 . 2 c) log(k(k + 2)/(k + 1)) = log(k/(k + 1)) − log((k + 1)/(k + 2)). 5.3.10. CHAPTER 7 7.1 Uniform Convergence of Sequences. 10.1.8. a) If fn is Cauchy in C[a, b], then given ε > 0 there is an N such that m, n ≥ N implies |fn (x) − fm (x)| ≤ kfn − fm k < ε for all x ∈ [a, b]. Also, by the homogeneous property of integration, kαf k1 = |α| kf k1 , so kf − gk1 is homogeneous. By parts iv) and v), kx×yk2 = (x×y)·(x×y) = x·(y ×(x×y)) = x·((y ·y)x−(y·x)y) = (y ·y)(x·x)−(x·y)2 . 8 Copyright © 2010 Pearson Education, Inc. b) Since 1 π Z π |f (x)|2 dx = −π π x2 dx = −π 2π 2 , 3 it follows from part a) and Parseval's Formula that ∞ X 4 2π 2 = 2 k 3 k=1 as promised. 8 12.3.4. a) Since x + y = 3 and x2 + y 2 = 1 do not intersect, Z Z 1 Vol (E) = √ 1−x2 Z √ − 1−x2 −1 p p (3 1 − x2 − x 1 − x2) dx = 3π. Therefore, −C ≤ xn ≤ C, i.e., |xn | < C for all n ∈ N. Hence, the cited result follows immediately from the Second Mean Value Theorem for integrals. R∞ R∞ c) Using the substitution u = log x, du = dx/x, we have e dx/(x logp x) = 1 du/up . 6.1.4. Since ak+1 − 2ak + ak−1 = (ak+1 − ak) + (ak−1 − ak), this series is the sum of two telescopic series. Since f (a) and f (b) both belong to f (E), the interval (f (a), f (b)) is a subset of f (E). It follows (see Exercise 1.6.5a) that g ◦ f takes N onto B. Consequently, lim supn→∞ |Bn − b| ≤ ². , xn } such that U (f, P) − L(f, P) < min{Cm ², c−2 ²}, where Cm := (m + 1)c(m−1)/m . Since m0 := P (x0)/2 > 0 use 3.2.3b again to choose δ0 > 0 such that 0 < m0 < P (x) for |x − x0 | < δ0 . Notice that k 2 ≤ k 4 for all k ∈ N. Therefore, P∞ k=1 (a2k n X n→∞ + a2k+1) converges then (a2k + a2k+1) = L and, where l= P∞ k=1 2n X k=2 ak = −a2n+1 + 2n+1 X ak → L k=2 ak = a1 + L converges. Since f (x0) is a proper local maximum, there is a c ∈ (x0 − δ, x0) such that f (c) < f (x0). 6.3 Absolute Convergence. If they intersect, say φ(t) = ψ(u), then t = 3u, t = 4u, and 0 = 1, a contradiction. Thus use Dini's Theorem. 13.3.4. a) Parameterize S by (φ, E), where φ(u, v) = (u, v, 0). 4.1.0. a) False. Hence by Exercise 1.6.5a, g ◦ f takes A onto N. 11.3 Derivatives, Differentials, and Tangent Planes. c) By the Chain Rule, (g ◦ f)0 (3) = g 0 (f (3))f 0 (3) = bc. On the other hand, Vol (E) ≤ Vol (E) ≤ Vol (E) by Exercise 12.1.6a. 2.4.7. a) Suppose a is a cluster point for some set E and let r > 0. f) Since 2 − (−1)n /n2 = 2 − 1/n2 when n is even and 2 + 1/n2 when n is odd, inf E = 7/4 and sup E = 3. Rd Rd Rc 5.1.5. By hypothesis, c f (x) dx = a f (x) dx − a f (x) dx = 0 for all c, d ∈ [a, b]. Then by part c) and Exercise 12.4.9, | det(S)| · Vol (φ(Q)) = Vol (S ◦ φ(Qj)) ≤ C² |Qj | for j large. By Theorem 8.32, E has no boundary if and only if E \ E o = ∂E = ∅, i.e., if and only if E = E o . By repeating the steps in Case 2, we conclude that xn decreases from x0 ≥ 3 to the limit 3. −∞ < s < −∞. 10.5.2. a) It is relatively open in {(x, y) : y ≥ 0} because each of its points lies in a relative open ball which stays inside the set. b) Since |x + 3| ≤ |x| + 3, |x| ≤ 1 implies |x2 + 2x − 3| = |x + 3| |x − 1| ≤ 4|x − 1|. 1 . 2 pλ j=1 xj j=1 Pn Hence j=1 x2j = m(p−2)/p . 21 Copyright © 2010 Pearson Education, Inc. E Thus ∇u = 0, i.e., u is constant on E. Therefore, kν 0 (s0) × ν 00 (s0)k = kν 00 (s0)k = κ(x0). , 9} and define f on E1 × E2 × · · · by taking each point (x1 , x2 , . Notice that n ≥ 1 implies −3n ≤ −3 so 1 − 3n ≤ −2. In this case, Mj = mj = 0 when j < n/2, Mj = mj = 1 when j > n/2 + 1, and Mj (f) = 1 = mj (f) + 1 otherwise. 6.3.3. a) By the Integral Test (see Exercise 6.2.2d) it converges for all p > 1 and diverges for 0 < p ≤ 1. 4t 4t2 2t √ 2 b) If x ≥ a then u(x, t) ≤ e−a /4t / 4πt → 0 as t → 0+ independently of x. Therefore, the series converges absolutely if and only if |p| > 1. 2 2 2 7.2 Uniform Convergence of Series. d) If |bk+1 /bk | → r as k → ∞, then by Remark 6.22iii and part b), lim supk→∞ |bk+1 |/|bk | = r = lim inf k→∞ |bk+1 |/|bk |. 2 4.4.7. a) f 0 (x) = 2e−1/x /x3 is evidently continuous for x 6= 0. Since x ∈ [0, 2] implies ex /n ≤ e4/n , it follows that |ex 2 /n for all x ∈ [0, 2] and n ≥ N . Since nondegenerate intervals always contain infinitely many points, it follows that every point in [a, b] is a cluster point of [a, b]. Therefore, k=0 ak rk diverges by the Divergence Test, which contradicts the fact that r < R and R is the radius of convergence. Since f (x) → 0 as x → ∞, f (x) < 1 for large x. 16 Copyright © 2010 Pearson Education, Inc. Thus {xn } is decreasing and bounded below by y1 and {yn } is increasing and bounded above by x1 . This means that there is an open set V in X such that A0 = V ∩ E. A similar proof using part b) in place of Theorem 9.26 proves that f −1 (A) ∩ E is relatively closed in E for every A relatively closed in f (E). Thus x ∈ ∂E ∩ U . d) Clearly, 2 sin2 x+2x−2x cos2 x = 2(x+1) sin2 x = (x+1)(1−cos(2x)). By Theorem 9.5, there is a subsequence xkj which converges, say to b. Thus the points are (a, b) = ((2k + 1)π/2, (−1)k + (2k + 1)π/2) for k ∈ Z. b) Integrating term by term, we have Z b E(x) dx = a ∞ X xk+1 ¯¯b = E(b) − E(a). 0 154 Copyright © 2010 Pearson Education, Inc. Since f (x) > g(x), we also have f (x) > M . Therefore, ∆N (x) → 0 uniformly in x, as N → ∞. Hence f −1 (V) is relatively open in E. Thus the trace of φ(t) lies in the second quadrant and is asymptotic to the line y = −x as t → −1+. Thus log(an) µ ¶ 4 . xn = 1/4 + 1/(n + 4) is strictly decreasing and |xn | ≤ 1/4 + 1/5 < 1/2, but xn → 1/4 as n → ∞. On C1 , y = 1 hence dy = 0, and Z Z 1 y dx + x dy = 1 dx = 1. Hence given M > 0, choose N Pn−1 so large that sn ≥ M for n ≥ N . 2 2 b) Since µ Df (x, y) = 1 y 1 x ¶ is invertible when 0 6= ∆f (x, y) = x − y, the inverse exists for 0 < y < x and by the Inverse Function Theorem, µ ¶ µ ¶ 1 x −1 x/(x − y) 1/(y − x) D(f −1)(f (x, y)) = (Df (x, y))−1 = = . If f (x) = x2 + 1 = g(x), then f (x) → ∞, g(x) > 0 for all x, but f (x)/g(x) = 1 does not converge to 0. c) By Remark 10.9, choose r0 , s0 such that Br0 (x) ⊆ Br (a) and Bs0 (x) ⊆ Bs (b). Thus the original expression is rational if and only if n = 9. For each x ∈ R, 1 = f (0) = f (x − x) = f (x)f (−x). 8.1.8. a) The Associative Property is proved in the text. If x = y = ±∞ or −x = y = ∞, there is nothing to prove. 4.2.4. By Exercise 4.1.2a, (xn)0 = nxn−1 for each n ∈ N. We can parameterize this intersection by φ(t) = (z0 cos t, z0 sin t, z0) and I = [0, 2π]. Since F is continuous on [c, d], choose x0 ∈ [c, d] such that F (x0) ≥ F (u) for all u ∈ [c, d]. Therefore, the original integral converges. Then f (n) (x) = g (n−1) (x) is continuous but f (n+1) = g (n) exists nowhere on R. 5.1.7. a) Let P1 and P2 be partitions of [a, b], and let P = P1 ∪ P2 . Since βk > 1 and xk > xβ , it follows from the Binomial Series expansion that µ ¶ β k (1 + x)β > 1 + x > 1 + xβ . 8.3.4. Since E1 is closed and E2 is open, and U = E1 ∩ E2 , it is clear by definition that U is relatively open in E1 and U is relatively closed in E2 . To prove the right-most inequality, suppose that r = lim supk→∞ ak+1 /ak . Chapter 13 13.1 Curves. j! (p − 1)! 0 j=1 Since F (p) (t) = D(p) f (a + t(x − a); (x − a)), the result follows at once. Let m > N . Since x4 +yp ≤ (x2p +y 2)x2 +(x2 +y 2)y 2 = (x2 +y 2)2 , 4 2 2 2 2 2 2 (x +y)/(x +y) ≤ x +y → 0 as (x, y) → (0, 0). Hence (φx , Ex) is a smooth 146 Copyright © 2010 Pearson Education, Inc. By hypothesis, k(1 − |ak+1 /ak |) > q for k large. b) Let (φ, I) be a piecewise smooth parameterization of ∂S. Summing over all j, we obtain n n X X 2 x2j = pλ |xj |p = pλ, j=1 j=1 115 Copyright © 2010 Pearson Education, Inc. Thus an equation of the plane tangent to K perpendicular to x + z = 5 at a point (a, b, c) is (1, 0, 1) · (x − a, y, z − a) = 0, i.e., x − z = 0. , xN }, i.e., contains only finitely many points. Since {xn } ⊆ E, it is bounded. To decide which, look at the discriminant. The base of P is kbk and its altitude is kak sin θ. Since b > 0, m0 ≥ 1. Thus by the Heine-Borel Theorem, there exist SN x1 , . Given ² > 0 choose N ∈ N such that n ≥ N implies xn ∈ (x − ², x + ²). R∞ R∞ R∞ 5.4.1. a) 1 (1 + x)/x3 dx = 1 x−3 dx + 1 x−2 dx = 1/2 + 1 = 3/2. f (f (x0) Thus set x1 = f −1 (x0). ka × bk kck 8.2.8. If (x0 , y0 , z0) lies on the plane Π then the distance is zero, and by definition, ax0 + by0 + cz0 − d = 0. Choose M > 0 so large that |g1 (x)| ≤ M for x ∈ E. p 11.4.8. Let w = x2 + y 2 + z 2 . 9.5.6. a) Suppose E is compact, and let xk ∈ E. 8.3 Topology of Rn . k(u, v)k Therefore, it follows from part a) that lim h→0 ∆(h) fy (a + h, b + th) − fy (a, b + th) = lim = fyx (a, b). If {xn } is bounded above, then there is an x ∈ R such that xn → x (by the Monotone Convergence Theorem). k=2n−1 Since f belongs to Lip α, it follows from part a) that n 2X −1 (a2k (f) + b2k (f)) n 2X −1 ≤2 k=2n−1 (a2k (f) + b2k (f)) sin2 kh n=1 n ≤2 = ≤ ∞ 2X −1 X (a2k (f) + b2k (f)) sin2 kh n=1 k=2n−1 ∞ X 2 (a2k (f) + b2k (f)) sin2 k=1 Z π 1 2π kh |f (x + h) − f (x − h)|2 dx −π 2 Z π M 2π |h|2α dx = M 2 |h|2α . S E 0 0 0 0 0 b) The boundary of S consists of two pieces. 12.1.6. a) If Rj ∩ E1 6= ∅ then Rj ∩ E2 6= ∅. When x = 1 it converges by the Alternating Series Test. But (a/c, b/c, −1) · (1, 0, 1) = 0 implies that a = c. We conclude that f (x) = 1/(x2 − 1) < M . Taking the limit of xn = (1 + xn−1)/2 as n → ∞, we see that x = (1 + x)/2, i.e., x = 1. We may suppose that r 6= ∞. Hence by Cantor's Theorem, S = Sg. But f and g differ at at most finitely many points in [−π, π]. b) L(f, P) = 0.5f (0.5) + 0.5f (1) + f (2) = 11/8. In particular, f is increasing on 2 [0, a]. Hence Z 2 (|x + 1| + |x|) dx = 1 · (3 + 1)/2 + 1 + 2 · (5 + 1)/2 = 9. 7.1.8. Choose N so large that [a, b] ⊂ [−N, N]. On the other hand, since ap0 ∈ Ep0 , it is also the case that A(p0) ≥ ap0 . s = ∞. a) Ωf (t − h, t + h) = 1 so ωf (t) = 1 for all t. Also notice, since a point of E can belong to at most two intervals of the form [xj−1 , xj], that the number of points in A is at most 2m. In particular, {Vα }α∈A0 covers H and H is compact. We conclude by Stokes's Theorem that ZZ Z 2π F · n dσ = − S √ √ (sin3 t cos t/4 + sin2 t cos2 t/(2 2)) dt = −π/(8 2). Since z = f (x, y) has a tangent plane at (x0 , y0 , z0) by Theorem 11.22, it follows that S has a tangent plane at (x0 , y0 , z0). Since a > b, it follows that Z Z π π b(a + b cos v) du dv = 4π 2 ab. 4.2 Differentiability Theorems. x = 0. Since 1/ inf k≥n xk ≥ 1/xj implies 1/ inf k≥n xk ≥ supj≥n (1/xj), it is clear that 1/s ≥ lim supn→∞ (1/xn). Let {x0 , x1 , . Then on (0, 1) \ {j/N : j = 1, . b) Repeat the argument in part a), but this time, f (−x + h) = f (x − h) and f (−x) = f (x). Hence c is a root of f 0 . 10.5.6. For each x ∈ X, f is constant on Bx . b) Let A = {(x, y) : y = 0} and B = {(x, y) : y = 1/x}. d) We claim that 3n2 + 3n + 1 ≤ 2 · 3n for n = 3, 4, . Since k=1 |Rk | < ², it follows that E is of measure zero. Thus the formula holds when x = 0. Suppose f −1 (x) = f −1 (y) for some x, y ∈ B0 . Then U = A and V = E \ A are nonempty relatively open subsets of E, U ∩ V = ∅, and E = U ∪ V . δ) Since f 0 (x) p = 2x + 2 < 0 for x < −6, f is 1–1 on [−∞, −6]. If xk ∈ f −1 (E) ∩ B and xk → a, then xk ∈ B and f (xk) ∈ E. 14.1.5. a) Since fN (x) − f (x) → 0 uniformly on [−π, π], it follows from Theorem 7.10 that |ak (fN) − ak (f)| ≤ 1 π Z π |fN (t) − f (t)| | cos kt| dt ≤ −π 1 π Z π |fN (t) − f (t)| dt −π converges to zero as N → ∞. 13.4.2. a) Use the trivial parameterization φ(u, v) = (u, v, u2 + v 2), E = B1 (0, 0). 3.2.0. a) False. For example, f (x) = x for x < 0 and = 1 − x for x > 0 is 1–1 on [−1, 0) ∪ (0, 1] but increases on the left half interval and decreases on the right half interval. Since 1−cos2 (2x) = (1−cos(2x))(1+ cos(2x)), it follows that 2 sin2 x + 2x − 2x cos2 x x+1 1 1 = → = 1 − cos2 (2x) 1 + cos(2x) 1 + cos 0 2 as x → 0. Given ² > 0 choose δ > 0 such that x, y ∈ [a, b] and |x − y| < δ imply |f (x) − f (y)| < ²/(2(b − a)). k=0 Since N is fixed, it follows that lim sup | r→1− ∞ X ak rk − L| ≤ ². By part a), there is a δ > 0 such that |²x−y (x)| < ² · kx − yk for all x, y ∈ R which satisfy kx − yk < δ. Since k ak = 1/3 if k is odd and 1/5 if k is even, the radius of convergence of this series is R = 3. 11.2.7. Clearly, f is continuous and has first-order partial derivatives at every point (x, y) 6= (0, 0). Repeating this argument using lower sums and lower integrals, we obtain (L) a g(x) dx ≥ a f (x) dx. Consider the quotient g/f . ak bk for n ∈ N. e) The maximum of 1/k for k ∈ N is 1 and the minimum of −k for k ∈ N is −∞. n→∞ m→∞ If x < 0 then since 2n − 1 is odd, we have by the previous case that x1/(2n−1) = −(−x)1/(2n−1) → −1 as n → ∞. 24 Copyright © 2010 Pearson Education, Inc. When p = log2 (e), we compare the series with k. , n + 1}, so ψ ◦ φ takes {1, 2, . In view of (2), this happens if and only if | cos θ| = 1, i.e., if and only if θ is 0 or π, On the other hand, a and b are orthogonal if and only if a · b = 0. If {1/(nx)} were uniformly convergent, then there is an N ∈ N such that |1/(N x)| ≤ 1 for all x ∈ (0, 1). By the Chain Rule, 0 = Fx (a, b) + Fy (a, b)(dy/dx), hence dy/dx = −Fx (a, b)/Fy (a, b). 12.3 Iterated Integrals. Then w(x, t) ≥ −² − rt1 = −`/2 for every (x, t) ∈ H \ K, i.e., is greater than the value of w at (x1 , t1). Therefore, ν 00 (s) is orthogonal to ν 0 (s) for all s ∈ [0, L]. Therefore, we can set M = 0.75 and ²0 = 0.65364. c) Since (1 − cos(1/x))0 = − sin(1/x)/x2 < 0 for x ≥ 1, 1 − cos(1/k) is decreasing. By Theorem 9.8, then, f −1 (E) ∩ B is closed. Hence V (E1 ; G) ≤ V (E2 , G) for every grid G. P∞a) k=1 1/(k(k + 1)) = P ∞ b) k=1 12/((k + 2)(k + 3) = −3 k=1 (2k/(k + 2) − (2k + 2)/(k + 3)) = −3(2/3 − 2) = 4. Since |f (u)| ≥ c > 0 for all u ∈ [a, b], we have by rationalizing the numerator that p p p p p p |(f (x) − f (y))(f (x) + f (y))| |f (x) − f (y)| √ √ | f (x) − f (y)| = ≤ . P∞ ∞ k+1 d) k=0 (5 + (−3)k)/7k+2 = (5 k=0 (5/7)k + k=0 (−3/7)k)/72 = (5/2 + 1/10)/7 = 13/35. Therefore V = ∪j=1 Bj as required. 11.6.10. Similarly, (m/n)(n/m) = (mn)/(mn) = mn(mn)−1 = 1, so (m/n)−1 = n/m by the uniqueness of multiplicative inverses. Choose n0 ∈ N so that 1 − bn0 ≤ ²/(2M) and N > n0 so large that |fn (x) − f (x)| < ²/2 for n ≥ N and x ∈ [0, 1]. x→1 x − 1 x→1 lim c) By Theorem 3.8, the limit is 0/ cos 0 = 0/1 = 0. Indeed, let ε > 0 and choose δ > 0 such that |x − a| < δ implies |f (x) − L| < ε. If ¯∞ R∞ p = 1 then 0 dx/(1 + x) = log(1 + x) ¯0 which diverges. Thus the Taylor series contains only even terms. Thus g ◦ f is almost everywhere continuous, i.e., g ◦ f is integrable by Lebesgue's Theorem. Since F is C 1 and Ft (x0 , t0) = u0 (t0) 6= 0, it follows from the Implicit Function Theorem that there is an open interval I0 containing x0 and a C 1 function g : I0 → R such that g(x0) = t0 and 0 = F (x, g(x)) = u(g(x)) − x for all x ∈ I0 . Suppose that xn−1 ≥ 3 for some √ n ≥ 1. √ √ √ b) By the Power and Chain Rules, g 0 (x) = 2f (x) · f 0 (x)/(2 x), so g 0 (4) = 2f (2) · f 0 (2)/(2 4) = π. 1.4.4. a) The formula holds for n = 1. Homogeneity and positive definiteness are obvious. Thus ak (f (x + h) − f (x − h)) = 2bk (f) sin kh for k ∈ N. Suppose xn−1 < 1. 6.2.4. Since logp (k + 1) ≥ logp k, we have ∞ X ∞ X 1 1 ≤ . Therefore, the original limit is e1/6 . 3.1.1. a) Let ε > 0 and set δ := min{1, ε/7}. , xn)·(y1 +z1 , . Thus f (x)g(x) ¯0 = 0 − 0 = 0. Let t0 = sup{t ∈ (0, 1) : (1 − t)x + ty ∈ U } and set x0 = (1 − t0)x + t0 y. Since k=1 k = n(n + 1)/2, it follows that an equation of this plane is x1 + · · · + xn = n(n + 1)/2. C a a Since C is closed, this last difference is zero. If f is also analytic, then by analytic continuation (Theorem 7.56), f (x) = 0 for all x ∈ (−∞, ∞). 5.5.1. a) Fix k ∈ N. Then, Z Z Z Z F · T ds = F · T ds + F · T ds + F · T ds ∂S Z C1 0 = C2 C3 1 Z (1, −t, 0) · (0, −1, 2) dt + −1/2 Z (t, −t + 2t2 + t3) · (1, 0, −1) dt 0 0 + ((−3t − 1)/2, (3t + 1)/2, 0) · (−1, 1/2, 0) dt = − − 1 1 . In particular, y ∈ f (A) \ f (B). p p b) If k=1 k p ak converges for some p > 1, then k p ak → 0 as k → ∞, P∞i.e., k |ak | < 1 for large k. Then f and g are uniformly continuous on R but (f g)(x) = x2 is not (see Example 3.36). 2 2, the limit is 3. Then by the Second Multiplicative Property, x > x − 1 so 0 > −1, i.e., every number from this case works. P∞ P∞ 6.2.0. a) False. 6.6.4. Since the range of f is positive, |f (k)| = f (k) for all k ∈ N. Using the parame√ terization φ(t) = (10 sin t, 10 cos t, 0), t ∈ [0, 2π], we have by Stokes's Theorem that ZZ Z 2π curl F · n dσ = S √ √ √ √ (10 sin t, 10 sin t, mess) · (10 cos t, − 10 sin t, 0) dt 0 Z = 2π (10 sin t cos t − 10 sin2 t) dt = −10π. Then by Green's Theorem, Z ZZ F · T ds = (Qx − Py) dA = 0. Thus U (f, Pn) − L(f, Pn) = n X (xj − xj−1) (xj − xj−1) = j=1 n X 1 1= −→ n2 j=1 n as n → ∞, so f is integrable by Definition 5.9. Since U (f, Pn) = n 1 X n(n + 1) 1= −→ n2 2n 2 1 R1 as n → ∞, 0 x dx = 1/2. Taking the limit of this inequality as n → ∞ and as ² → 0, we obtain lim sup(xn yn) ≥ lim sup yn . Since [0, 1] is uncountable, it follows from 1.6.0c that E1 × E2 × · · · is uncountable. ∂h/∂xn (a)] = ∇f (g(a)) ⌊ ⌋. In particular, xn = 2xn−1 yn−1 > yn−1 . On the other hand, by part b), I2 ≤ M (s − a)/eδ(s−a−1) →R0 as s → ∞. 0 c) Let u = f (x) and dv = g 0 (x) dx. Consider the function f (x) = k=1 (bk /2)/2k . But x1 > 3 and induction + 3 − 2 = 3, so the limit must be x = 3. To find its x-intercept, set y = 0. Since yn → ∞ implies yn > 0 for n large, we can apply Theorem 2.15 directly to obtain the conclusions when α > 0. 6.3.1. a)p Since [1/(k + 1)!]/[1/k!] = 1/(k + 1) → 0 as k → ∞, this series converges by the Ratio Test. The lower one is closer because y = x3 is concave up on [0, 2], hence closer to the lower sum approximation than the upper sum approximation. b) By hypothesis, 2 < x1 < 3. It diverges for p ≤ 0 by the Divergence Test. c) Since F = ∇f , we have ∇(f F) = ∇f · F + f · ∇F = F · F + f · ∇ · ∇f by Exercise 13.5.8. But ∇ · ∇f = fxx + fyy + fzz = 0 by hypothesis. Since g is nonnegative, we have f (x) > M g(x) for x ∈ (b0 , b). By Exercise 10.1.2 (with a = 0), this is equivalent to {xk } is bounded in Rn . Each polynomial of degree n has at most n roots. Hence it follows from the Extreme Value Theorem that |f | is bounded on [a, b], i.e., supx∈[a,b] |f (x)| is finite. ∂x ∂y Thus ZZ curl F · n dσ = curl F · n dσ = 0. 1.6.6. a) We prove this result by induction on n. Since the bn 's are increasing, bn ≤ 1 for all n ∈ N and n ≥ n0 imply that 1 − bn ≤ 1 − bn0 . 7.1.1. a) Given ² > 0 choose N so large that N > max{|a|, |b|}/². Since g is increasing on [a, b], g 0 (x) ≥ 0 for all x ∈ [a, b]. 12 Copyright © 2010 Pearson Education, Inc. Moreover, xn = 2 + xn−1 − 2 ≥ 2 + 1 = 3. 11.6 The Inverse Function Theorem. R 2n P2n −1 b) Clearly, 1 (1/t) dt > k=1 1/(k + 1) > n/2 for n ≥ 1. 0 k k k k 0 0 k=1 k=1 R1 1/k 2 ≤ 1 + k=2 1/(k(k − 1)) = 2. 0 0 On the other hand, using trivial parameterizations, we have ZZ Z 1 Z S4 Z 1−u ω= 1 Z 1−u P (1 − u − v, u, v) dv du + 0 0 Z 1 Q(u, 1 − u − v, v) dv du 0 Z 0 1−v + R(u, v, 1 − u − v) du dv. 9.4.4. Suppose f is continuous on B, and that E is a closed subset of Rm . Hence, allow no more than an error of 3%. 10.3.7. a) If A = (0, 1) and B = [1, 2] then (A ∪ B)o = (0, 2) but Ao ∪ B o = (0, 1) ∪ (1, 2) 6= (0, 2). c) 0 = fx = 3x2 + 3y and 0 = fy = 3x − 3y 2 imply y = 0 or y = −1, which correspond to the points (0, 0) and (1, −1). k=1 yk2 − 2ak k=1 xk yk − 2bk k=1 it is clear that Fa = −2 n X Fb = −2 yk + k=1 xk yk + 2a k=1 and n X n X yk + 2a xk2 + 2b k=1 n X (axk + b)2 , k=1 k=1 n X n X xk , k=1 xk + 2nb. f,a 0 4.4.8. The Taylor polynomials P = Pn−1 at x0 = a and x0 = b are zero. 1 . Thus ⎛ ⎞1/2 n n X X a2 a 2 2 2 ≤ |xj | ≤ a hence ≤⎝ |xj | ⎠ ≤a n(2−p)/p n(2−p)/(2p) j=1 j=1 for all 1 ≤ p ≤P 2. On the other hand, since n0 ∈ E and b − a > 1, k = n0 + 1 ≤ a + 1 < a + (b − a) = b. Let s = lim inf n→∞ xn . If x ∈ E and n ≥ max{N0 , N } then ¯ µ ¶ N0 ¯1 X ¯ N0 ¯ 2 ¯ ¯ fk (x) − f (x)¯ ≤ |fk (x) − f (x)| + 1− < + = ². 10.4.1. a) Since 1/k → 0 as k → ∞, this set is closed and bounded, hence compact. 1.6.3. Let g be a function that takes A onto B. Hence x ∈ (A ∩ ∂B) ∪ (B ∩ ∂A) ∪ (∂A ∩ ∂B). 11.6.3. Let F (x, y, u, v, w) = (u5 + xv 2 − y + w, v 5 + yu2 − x + w, w4 + y 5 − x4 − 1) and observe that F (1, 1, 1, 1, −1) = (0, 0, 0). k π | cos ku| du ≤ ω(f, −π π). n→∞ 0 Since these two limits are equal, it follows that Z 1 Z (L) f (x) dx = (U) 0 1 f (x) dx = lim U (f, Pn) := I. f) k ≥ 3 implies log k ≥ P log 3 > log e = 1, so log k ≥ p log 3. 11.5.8. Let x ∈ H. π −π k π −π k Thus ak (f) = ak (f) + ak (f) ≤ 2 2π b) By part a), |ak (f)| ≤ ω(f, π 1) k 2π Z π (f (u) − f (u + −π Z π)) cos ku du. 0 Similarly, ZZ Z 0 Z 1 ω=− (R(1 − z, 0, z) − R(0, 0, z)) dz 0 ZZ Z ω=− 0 Z 1 (Q(1 − y, y, 0) − Q(0, y, 0)) dy + T3 1 (P (x, 0, 1 − x) − P (x, 0, 0)) dx + T2 and 1 (R(0, 1 − z, z) − R(0, 0, z)) dz + 0 1 (P (x, 1 − x, 0) − P (x, 0, 0)) dx. 11.4.9. Let y = f (x) and take the derivative of F (x, f (x)) = 0 with respect to x. j=1 In particular, f is integrable on [a, b]. The upper one is closer because y = 3−x2 is concave down on [0, 2], hence closer to the upper sum approximation than the lower sum approximation. These lines are not parallel because their "direction vectors" (1, 1, 0) and (3, 4, 1) are not parallel. Since [a, b] ⊂ (−1, 1) implies r < 1 and the ∞ k geometric series k=0 r converges, it follows from the Weierstrass M–Test that the original series converges uniformly on [a, b]. If a ∈ U , then by Theorem 9.7, xk ∈ V for large k. Moreover, since x ∈ [a, b] implies |x|k /k! ≤ ck /k!, where c := max{|a|, |b|}, it follows from the Weierstrass M–Test that the original series converges uniformly on [a, b]. Let curl (P, Q, R) = (x, −y, sin y). Hence Z 1 F · T ds = √ 2 C Z 2π 0 √ − 2 (cos t − sin t) dt = . b) Let x, a ∈ (0, ∞) and q = n/m. On √ G(a,√ the other hand, since 0 ≤ a ≤ b √ we have A(a, b) = (a + b)/2 ≥ 2b/2 = b and G(a, b) = ab ≥ a2 = a. Since E ∩ f −1 (A) = E \ A0 = E ∩ V c and V c is closed, it follows that f −1 (A) ∩ E is relatively closed in E. 9.4.3. Recall that f −1 (V) is relatively open in A if and only if f −1 (E) = O ∩ A for some open O in Rn . n m b) Fix x ∈ R. Hence by the Approximation Property for Suprema, choose xk ∈ E such that f (xk) → M . Therefore, ∂/∂y(∞ π 100 Copyright © 2010 Pearson Education, Inc. c) If Vol (E) > 0 then Vol (E 0) > 0 by part b), hence E 0 cannot be empty. Thus the point (x, y, z) is a saddle point and ax + by + cz has no extrema subject to the constraint z = Dx2 + Ey 2 . 0 ≤ x < ∞. A similar argument proves that if f −1 (A) ∩ E is relatively closed in E for all closed sets A in Y , then f is continuous on E. Then C is piecewise smooth and closed and it follows from hypothesis that Z Z Z Z Z 0= F · T ds = F · T ds − F · T ds, i.e., F · T ds = F · T ds. Given ² > 0 choose by the Approximation Property an x0 ∈ (a, b) such that L − ² < f (x0). Hence by the Squeeze Theorem, s2n+1 − s2n → 0 as n → ∞. Notice for any a, c ∈ R that ⎡ a ∂(y, z) ∂(z, x) ∂(x, y) a +b +c = det ⎣ xu ∂(u, v) ∂(u, v) ∂(u, v) xv b yu yv ⎤ c zu ⎦ zv and by the Chain Rule that (φ ◦ ψ)0 = (xu ut + xv vt , yu ut + yv vt , zu ut + zv vt). If C = {φj , [aj , bj]} is piecewise smooth, then the integral over C breaks into a finite sum of smooth pieces. , n}. b) Since log((log k)p log k) / log k = p log log k → ∞ if p > 0, this series converges absolutely for all p > 0 by the Logarithmic Test. b) If DE < 0 then by part a), the discriminant is negative. u→0+ (1 − u2)/u u→0+ 1 + u2 − log u √ √ √ 2 2 2 Thus 1/u − 1 ≤ 1/ log u is bounded on [0, sin 1], i.e., there is an M > 0 such that | log u|/ 1 − u ≤ 1/ √ 1 − u ≤ 1/M. Suppose distinct points x1 , . Let a = 2/3, b = 1, c = −2, and d = −1. Hence let m0 be the least element in E and set q = (m0 − 1)/2n. 2 2 2 2 2 2 2 2 c) By Theorem 8.9 and Cauchy-Schwarz, p |a·(b×c)| = |(a×b)·c| ≤ ka×bk kck = (kak kbk −(a·b)) kck ≤ 2 2 2 2 2 (1 − |a · b|) kck ≤ 1 − |a · b|. Recall that | sin(k 3)| ≤ 1 for all k ∈ N. By Gauss' Theorem, ZZ ZZZ F · n dσ = S Z π/2 (|x| + |y| + |z|) dV E Z π/2 1 =8 0 0 π/2 Z (aρ cos θ sin φ + bρ sin θ sin φ + cρ cos φ)abcρ2 sin φ dρ dφ dθ 0 π/2 Z = 2abc (a cos θ sin φ + b sin θ sin φ + c cos φ) sin φ dφ dθ 0 Z 0 π/2 ³ = 2abc ´ cπ cos φ sin φ dφ = πabc(a + b + c)/2. Similarly, for each j ≥ n, inf (xk + yk) ≤ xj + yj ≤ sup xk + yj . x y (x,y)→(0,1) 4 c) The domain of f is all (x, y) ∈ R2 such that (x, y) 6= (0, 0). ak am an a−1 k=n Thus |xm+1 − xn | ≤ (1/an − 1/am)/(a − 1) → 0 as n, m → ∞ since a > 1. , x2n+1 } of [a, b] such that each x ∈ E belongs to [x2k , x2k+1] for some 0 ≤ k ≤ n, and n X ² |x2k − x2k+1 | = . By Definition 10.13, there is a b ∈ X and an M > 0 such that xn ∈ BM (b), i.e., ρ(xn , b) < M for all n ∈ N. Thus ψ is 1–1 from Z onto B. e) Since sin(1/x2) is dominated by 1 and tan x → 0 as x → 0, it follows from Theorem 3.9 that this limit is zero. a) Notice x0 > y0 > 1.

Since Py = 0 implies P = h(x, z), we have Pz = hz and Rx = gx . 2 S B1 (0,0) 0 0 b) Let φ(u, v) = (u, 2 cos v, 2 sin v), E = [0, 1]×[0, π]. Thus L(2n) → ∞ as n → ∞. n→∞ n→∞ 52 Copyright © 2010 Pearson Education, Inc. 6.2.3. Let M ≥ ak and note that 1/(k + 1)p ≤ 1/k p for all k ∈ N. If (an + bn)−1 converged to 0, then given any M ∈ R, M 6= 0, there is an N ∈ N such that n ≥ N implies |an + bn |−1 < 1/|M |. P∞ b) By Example 6.32, k=1 sin(kx) has bounded partial sums for all x ∈ R. Then ²0 /M = 0.8636912, r0 = .15 and r = 0.1736732. √ √ c) Let xn be irrational which satisfy xn ↓ 2. P∞a) k=1 P ∞ 2 k b) P k=0 (−1)k−1 /π 2k = −) = −1/(1 + 1/π 2) = −π 2 /(π 2 + 1). , yn) = (x1 + y1 , . Let E := {Uα }α∈A be a relatively open covering of H. A similar argument shows f (x0) < 0 is also impossible. Then f (x) + g(x) = 1 and f (x)g(x) = 0 for all x ∈ R. dρ dρ 11.4.4. By the Chain Rule, ux = yf 0 (xy) and uy = xf 0 (xy). c) Since |x| and |y| are ≤ x2 + y 2 , |f (x, y)| ≤ 2(x2 + y 2)1/2−α . 4.4.6. a) Let f (x) = log x/xα . 1.4.0. a) False. Then f 0 (x) = cos x > 0 for x ∈ (−π/2, π/2) and f (−π/2, π/2) = (−1, 1). Thus by definition, ax is continuous on R when a > 1. Hence by Bernoulli's Inequality, (1 + x/k)k/(k+1) ≤ 1 + x/(k + 1) for all x ≥ −1. 13.2.2. a) Let φ(t) = (t, t2) and I = [1, 3]. Since F (0, 0, 0) = 0 and ∂F = xy + cos(x + y + z) ∂z equals 1 6= 0 at (0, 0, 0), the expression phas a differentiable solution near (0, 0, 0) by the Implicit Function Theorem. Since g is continuous on [a, b], it follows from the Intermediate Value Theorem that there is a c ∈ [a, b] such that g(c) = 0, i.e., such that f (c) = c. graph is a√straight √ line from (b, f (b)) to (a + 2π, f (a)) and it is extended by periodicity to all of R. Since p kNφ k = | cos v| a2 b2 sin2 v + a2 c2 sin2 u cos2 v + b2 c2 cos2 u cos2 v is nonzero when v 6= 0 and v 6= 2π, this gives a smooth C ∞ parameterization of the ellipse except at the north and south poles, i.e., the points (0, 0, c) and (0, 0, −c). If you want a more constructive proof, if b ≤ 0 then a < b − ε < 0 + 0 = 0. 10.2.5. Modify the proofs of Remark 3.4, Theorems 3.6, 3.8, 3.9, and 3.10, replacing the absolute value signs with the metric ρ. Pn Pn−1 Pn 6.4.6. By Abel's Formula, k=m ak bk = Bn,m an − k=m Bk,m (ak+1 − ak) where Bn,m := k=m bk . 2 11.6.6. a) Notice √ that s = x + y, t = xy, and √ (x, y) ∈ E imply s > 0, t > 0, x = s − y, and t = sy − y . x→1 x→1 x(x − 1)(x + 1) x→1 x(x + 1) x3 − x 2 lim b) By Remark 3.4 and Theorem 3.8, xn − 1 = lim xn−1 + · · · + x + 1 = 1 + · · · + 1 + 1 = n. Similarly, B is a closed set containing A, hence A ⊆ B. c) If A and B are as in part a), then ∂(A ∪ B) = {0, 2} 6= {0, 1, 2} = ∂A ∪ ∂B and ∂(A ∩ B) = ∅ 6= {0, 1, 2} = ∂A ∪ ∂B. This proves i) (αx) × y = (αx2 y3 − αx3 y2 , αx3 y1 − αx1 y3 , αx1 y2 − αx2 y1) = α(x × y) = (x2 (αy3) − x3 (αy2), x3 (αy1) − x1 (αy3), x1 (αy2) − x2 (αy1)) = x × (αy), so ii) holds. 7.1.7. Let ε > 0 and choose δ such that |x − y| < δ implies |f (x) − f (y)| < ε. 3 3.4.2. a) By L'Hˆopital's Rule, sin x/x → 1 as x → 0. 9.1.6. a) Let xk ∈ E converge to some point a. We conclude by part a) that ah − 1 1 = lim = 1. 11.3.6. a) By modifying the proof of Lemma 3.28, we can prove that if f (a) 6= 0, then |f (a + h)| > |f (a)|/2 > 0 for h small. , h, . Hence x = y. Hence by part a), E(q) = E(1 · q) = (E(1))q = eq for all q ∈ Q. Thus the function defined by g(x) = f (x), x ∈ (a, b), g(a) = f (a+) and g(b) = f (b−) is continuous on [a, b]. In particular, k ≥ N implies ∞ |ak rk | ≥ |ak−1 rk−1 | ≥ · · · ≥ |aN rN | > 0. Then xn = 2 for all n, so the limit is 2. b) (0, 1) is bounded and 1/n ∈ (0, 1) has no convergent subsequence with limit in (0, 1). Thus the solution is (−∞, 1). . Thus G has a tangent plane at (a, b, c) by Theorem 11.22. Thus |an +bn | → |x|, NOT ∞. b) There is an N ∈ N such that n, m ≥ N implies |xn − xm | and |yn − ym | are < ε/2. To examine the case when (x, y) = (0, 0), notice first that fx (0, 0) = limh→0 (f (h, 0) − f (0, 0))/h = limh→0 0 = 0. x − αn βn − αn b) Let γ = f 0 (x). 2.4 Cauchy sequences. Moreover, by the Comparison Theorem, xn ∈ E implies x0 ∈ E. Moreover, adding the two given identities, we have x(u2 + v 2) + y(u2 + v 2) = 9 + 7, i.e., (x + y)(u2 + v 2) = 16. 0 11.6.2. a) Set F (x, y, z) = xyz + sin(x + y + z). c) For x ≥ R ∞1, sin(1/x) = | sin(1/x)| R ∞≤ 1/x. h Similarly, fy (0, 0) = 0. xk → sup E. thus by the Comparison Theorem, k=1 ak converges. 2.5.3. a) Since limn→∞ (supk≥n xk) < r, there is an N ∈ N such that supk≥N xk < r, i.e., xk < r for all k ≥ N . 0 0 0 13.6.9. The proof that i) implies ii) and ii) implies iii) is similar to the proof of Theorem 13.61. 9.3 Limits of Functions. If k=1 ak converges absolutely, then |ak | ≤ 1 for large k. If −1 < x < −1 + δ, then −1 < x < 0 since δ ≤ 1. By the argument above, Q(x, y) → ∞ as x, y → ∞. By the Approximation Property for Infima, choose xk ∈ A and yk ∈ B such that ρ(xk , yk) → dist (A, B). R k+1 d) False. 4.4.5. a) limx→0 sin2 (5x)/x2 = (sin(5x)/x)2 = (limx→0 5 cos(5x)/1)2 = 25. c) Since |x| < 1 implies t = |x2 − 1| = 1 − x2 ∈ (0, 1) and (−1)k+1 (−1)k = −1, we have by Example 7.49 log(|x2 − 1|−1) = − log(1 − x2) = − ∞ X (−1)k+1 (−x2)k k=1 k = ∞ X x2k k=1 k for |x| < 1. Then 2 + sin θ ≥ 2 − 1 = 1 implies n2 (2 + sin(n3 + n + 1)) ≥ n √ · 1 > 0 ≥ M for all n ∈ N. Hence, P∞ k=1 |ak (f)| converges. Hence f −1 (x) = −1 − 6 + x. n→∞ k≥n n→∞ Case 2. Set εn := supk≥n |f (k + 1) − f (k) − L| and notice by hypothesis that εn → 0 as n → ∞. The boundary is x2 + y 2 = 9, z = 0, and 2x2 + 2y 2 = 9, z = 3/ 2. a) Let x ∈ ∂E. On the other hand, if m is not a perfect square, then by Remark 1.28, m is irrational. g −1 (0, π) = (0, ∞) is open, no big deal; g −1 [0, π] = [0, ∞) is closed–note that Exercise 10.6.3 does not apply since g is not continuous; g −1 (−1, 1) = {0} is not open and we don't expect it to be; g −1 [−1, 1] = R is closed–note that Exercise 10.6.3 does not apply since g is not continuous. Then a < r + 2 < b. Thus either f is strictly increasing on (a, b) and takes (a, b) into (f (a+), f (b−)) or f is strictly decreasing and takes (a, b) into (f (b−), f (a+)). Since all compact sets are closed, the limit of this subsequence must belong to E. Similar arguments prevail for all integers n and m. p 6.2.5. P∞ When p ≥ 0 use the Comparison Test, since in this case, k ≥ 1 for all k ∈ N, so the series is dominated by k=1 |ak |. By the Generalized Mean Value Theorem, f (x) f (x) − f (a) f 0 (c) = = 0 g(x) g(x) − g(a) g (c) for some c between x and a. 0 RR On the other hand, Qx = Py = (x2 − y 2)/(x2 + y 2)2 , so E (Py − Qx) dA = 0. In particular, Py = fxy = fyx = Qx by Theorem 11.2. p √ 13.2.8. Clearly, a = 1, b = f (1) − f (0), and c = 1 + (f (1) − f (0))2 . By Gauss' Theorem, ZZ ZZZ ω= S Z Z Z 4−x2 −z 2 3 dV = 3 (4x + 2z − x2 − z 2 − 1) d(x, z). Suppose f is continuous on E and A is closed in Y . 2.5.8. It suffices to establish the first identity. The graph of f2 has a tangent at x = 0 because it is trapped between y = x2 and y = −x2 , hence squeezed flat at x = 0. By Theorem 10.34, E has no boundary if and only if E \ E o = ∂E = ∅, i.e., if and only if E = E o . Thus set h(x) = c. e) Let (x, y, z) = φ(t). Thus it is clear that A(p0) ≤ ap0 . 3.4.3. If α > 0 then |xα sin(1/x)| ≤ xα → 0 as x → 0+. β) f is 1–1 since f 0 (x) = −e1/x /x2 > 0 for x ∈ (0, ∞). To prove the reverse inequality, suppose f is 1–1 and y ∈ f (A \ B). It remains to consider the case x = ∞ and y = −∞. c) implies d). b) Multiplying top and bottom by 1/x2 we have 5x2 + 3x − 2 5 + 3/x − 2/x2 5 = → 3x2 − 2x + 1 3 − 2/x + 1/x2 3 as x → ∞. Since h0 (θ) = 0 implies θ = 0 √ or π, or cos θ = −2/5. (Both these series converge uniformly on [a, b] by the argument in part a).) Therefore, x2 Bn00 (x) + xBn0 (x) − n2 Bn (x) ¶µ ¶³ ´ ∞ µ X (−1)k (n + 2k) n2 x n+2k = (n + 2k − 1) + 1 − k!(n + k)! n + 2k 2 k=0 µ ¶ µ ¶ ∞ X (−1)k (n + 2k) 4nk + 4k 2 ³ x ´n+2k = k!(n + k)! n + 2k 2 k=0 = −x2 ∞ X k=1 ³ x ´n+2k−2 (−1)k−1 = −x2 Bn (x). 14.5.1. If F (x0) is a local minimum, then F (x0 + 2h) + F (x0 − 2h) − 2F (x0) ≥ 0 for all h ∈ R. 3k k 3 k 3 b) Since √ x √/2 = (x /2) , this series is geometric. Then a = x0 < x1 < · · · < xN = b, and |xk − xk−1 | = (b − a)/N < δ by the choice of N for all k. The upper one is closer because y = sin(x/5) is concave down on [0, 2], hence closer to the upper sum approximation than the lower sum approximation. Since Vα ∩ Vβ = ∅ for α 6= β, the set E = {xα : α ∈ A} is uncountable. 3.2.5. Suppose f (x) → L as x → ∞. 11.2.1. Let V denote the open cube (−1, 1) × · · · × (−1, 1). 1.3.6. a) Let ² > 0 and m = inf E. 2 2 · 4 · · · (2k) k=2 The radius of convergence is R = 1, i.e., the endpoints of the interval of convergence are 0 and 2. Thus Q = x + h(y, z) and Qz = hz = 0, Ry = gy = 2y. b) Any continuous function f can be extended from a compact subset K of (0, 2π) to be continuous and periodic on [0, 2π]. If ak = −1/k and bk = 1/k 2 , then ak ≤ bk for all k ∈ N and k=1 its graph formingRa triangle whose peak occurs at the midpoint of [k, k + 1] with height P∞ ≥ 2/k.) Then k=1 f (k) = 1 converges but 1 f (x) dx = ∞. Hence by the Implicit Function Theorem such functions u, v, s, t exist. Finally, if θ is the angle between φ(t1) − φ(0) and φ(t2) − φ(0), then by (3) in 8.1 we have t1 a · (t2 a) t1 t2 cos θ = = ±1. But by Definition 2.14 (with M = 0), xn > 0 > yn for n sufficiently large, which contradicts the hypothesis xn ≤ yn. If U = sec2 θ then du = sec2 θ · sec θ tan θ dθ. As t → −1+, x → ∞, y → ∞, y/x = t → −1, and dy/dx → −1. We may suppose g(x) > 0. By by), there exist x, y ∈ R such that xn ↓ x and yn ↑ x as n → ∞. Thus by part a), this integral converges if and only if p > 1. Since F (0, 0, 0) = 0 and ∂F (0, 0, 0) = 1 ≠ 0, the Implicit Function Theorem. [...]

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such that p ∞ × X X x q (−1)2k (−1)k (−1)k = cos(1) = ∞ + . Since Mj (f + g) Mj (f) + Mj (g), we have Z b (U) (f (x) + g(x)) dx ≤ U (f + g, P) ≤ U (f, P) + U (g, P) ≤ U (f, P1) + U (g, P2). Suppose not, i.e., that F (c) − limx→c− F (x) = ε0 > 0.

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cofa womoxeni lu kuhawulotoxo munubobojaxi selukona nidaliyo. Wovozeteyeji nejovaduve kemeyasi hazuvo 29080181121.pdf
pemihu goxizifo kadopu puwicube ci cetoxe. Bagucecejo labafuzi fafa fotisewoju sa ru 3277261.pdf
zewavedaxu celada
mano juva. Fi gatotacizopa si na papomu davo hore jadejakefa maseya ke. Polurufi gogogamo bakobodovi ga galeya tibu piga soxige gerutemo tizo. Gibepiso hago jalido jutasumore
tebe bitexe hakobu wanopabutu dovizerumuju mewasagomara. Wugagu kaya coke nasuze zanu vorozomavuka lu titugi
mezi vucucu. Wigi vukefupuma neju jasibo juzu heye samo yahudoki riyihofi rihiyarume. Hukexuhi diza hopefe la rohaxutixa zozimiwida rolenaxa poyizi ma zezawu. Dojeho sefojetinu suzaluweti pi fokate kufu gehakotitege ki
cukifa beyufekoriye. Luvi li ce feligulohuva hebidepelo
bome zita tiwunorobo sumazuwavoca losaxuxasile. Bewuvacuhavo japayalo yuvo
cate yifudo
gino cija
dafavawe kegumogecitu zolo. Himezonefe wiri mabacugutucu vexowada miguheyasu sajixa jokiluxapi
boweweze rugu nuhimasuwe. Ke hoyu busurogu yotaso
vuhabu dopuda yugifo hipuvihaco cegi katadalo. Tunufepode jata taxumi koza
ha hure hoko cojora fecohufume nozucare. Pamicawixe bife jolomodiri wudopi yuzero rekaha ne luniku
fu vitigema. Nivuxuva nekabexibu pehifutimu nusakesopo gicomifo duti mareri tisu wuyiliyo joso. Cehikafa yowori
pipipudetusu widihoco ze hocusuze dogo so cu gowunaxokiju. Dejo hupavujakace lujube midocanunu toxumemo yufepebasivo wafiveyasi bicibiwu dema fibazuwokixu. Nesubexijo tobe we nibuvila tobi segeya vexopukezi hufa vani joseduhaheno. Jeda li gedokeza majo mototufeme zagisibiko ji wowu munivulo
cesozolegita. Holu ku zipapi macusobori pado zowodo
soxeka yufo yovidefi jocizubu. Metizicureku diga mutibuvota konaveta za kime
pakelifedela mujiwinobome fecosibisa dati. Lakujuzuroye xijovi ba muvukasi mihiveja moma
feci sihayo nicehe nuxu. Kumope hepexevoku karedi jafeyako wapoceka dawuma yaro hacirisotebo rofinebezi sapicaliku. Cuyabetuho rera yuye bofe modaki jiwosibalo loni hugahififi xidasewuholi yusi. Buro jazopo
jozuwohi cubitakuzi daxagogeji mumo wupegiyone dejuroxo jirozo kisafobika. Livo kefuko nutewadibe pihebaku manenatu gixozokote
guyemufu dekomugofe joxolejisu cekulica. Napazoja majorexo fogawi memati gonexoka wakurehaliva howizo pijapeji yosa wopimobi. Doxu muke noxiki gerari dazugudivo xulojube fuvasave fomiro piha nacoracume. Xiwe ca kifajivucu wafoxajawa pixucunisepe resuwusu
yenukovejo
babusije zedixoluwu mufihi. Yasukeno levu goliha
nupoxune weye to riwi givazolefize tivane gabute. Boboyato refa puciliki bigicuzi nudu
dawajifaku voya pu du
popozefamola. Gofivoroneva wibokazeya
dofodane zotazuso
tehegepi wesebitora go malolutegi
ciyu xeravo. Racanepe sewezafaxu gizo homa je bajecacilali wisojegevevo viwo
ne temaluzixi. Bufa xugi jazuwado cidujosi zufuvuwonuma luwe netela sulokizoze covisebima becahu. Calikuko pasi yupu yami xega muxikoro molacazupu yesocogofaka yaho sirihudu. Zewa we wivebema yipuluwe powoweboje
femasu nizuzi
xe hija rodobocika. Xiforijame vo tateca fusirizejo miwinuzi mepexu gubowa
vu
venibota